Section 13

1. Consider again the machine from Recitation 12, Problem 4. If the machine is working (state 0), the time until it breaks down is exponentially distributed with rate $\lambda$. If broken (state 1), the time to repair is exponentially distributed with rate $\mu$. Formulate the process as a uniformized CTMC, and find the transition matrix of the embedded DTMC.

2. For the machine in Problem 1, find $P_{00}(10)$, the probability that the machine will be working at time $t = 10$, given that it is functioning at time $t = 0$. This time, use uniformization.

3. Potential customers arrive at a full-service, two-pump gas station at a Poisson rate of 12 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended) at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of 10 minutes.

   (a) What fraction of the attendant’s time will be spent servicing cars?
   (b) What fraction of potential customers are lost?

4. Consider an $M/M/1$ queue. That is, interarrival and service times are exponentially distributed with rates $\lambda$ and $\mu$, respectively, and there is a single server. Assume $\lambda < \mu$. Find the limiting distribution and the long-run average number of jobs in the system.

5. One fairly well-known (and useful!) result from queueing theory is Little’s Law. For a queueing system that has reached “steady state”, Little’s Law states that

$$L = \lambda W$$

where $L$ is the average number of jobs in the system (both in queue and in service), $W$ is the average time that a job spends in the system, and $\lambda$ is the arrival rate.

This is true for general queueing systems; we do not need to assume that interarrival and service times are exponential, or that the queueing system is structured in a particular way.

   (a) For the gas station from Problem 3, how long, on average, do customers spend in the system? Does this match the value we would have obtained by working with the limiting distribution?

   (b) Find the average time that a job in an $M/M/1$ queue spends in the system, for the case $\lambda < \mu$. If $\lambda > \mu$, can we still use Little’s Law?