Section 6

Materials: important continuous random variables.

1. You arrive at a bus stop at 10:00 am, knowing that the bus will arrive at some time uniformly distributed between 10 am and 10:30 am.
   (a) What is the probability that you will have to wait longer than 10 minutes?
   (b) If at 10:15 am the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

2. (a) Show that for any $x > 0$ and $t > 0$,
   $$\mathbb{P}(X \geq x) \leq e^{-tx}\phi_X(t).$$
   (b) Suppose $X \sim \text{Gamma}(\alpha, \lambda)$. Recall from the lecture that for $t < \lambda$,
   $$\phi_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}.$$
   Show using (a) that
   $$\mathbb{P}(X \geq 2\alpha/\lambda) \leq (2/e)^{\alpha}.$$

3. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = e^X$. Compute $\mathbb{E}Y$ and $\text{Var} Y$. (Hint: use your knowledge of MGFs.)

4. Projectiles are fired at the origin of an $xy$ coordinate system. Assume that the point which is hit, say $(X, Y)$, is a pair of independent standard normal random variables.
   (a) Find $\mathbb{P}(X^2 + Y^2 \leq 1)$ without calculating integrals.
   (b) For two projectiles fired independently of one another, let $(X_1, Y_1)$ and $(X_2, Y_2)$ represent the points which are hit and let $D$ be the distance between them. Find the distribution of $D^2$.

5. A post office has two clerks. Service at clerk $i$ is $\text{Exp}(\lambda_i)$ distributed. You now arrive and require service, but both clerks are busy with one customer each, and apart from these customers there is nobody else in front of you waiting for service. Among you and the two previous customers, what is the probability that you are not the last to leave service?