Section 5

**Materials:** Correlation; moment generating function, Laplace transform, characteristic function; Law of Large Numbers, Markov’s Inequality, Chebyshev’s Inequality; discrete distributions.

1. Let $X$ and $Y$ be independent standard normal random variables, that is, they both have pdf given by

$$f_X(t) = f_Y(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad -\infty < t < \infty$$

Let $U = X + Y$ and $V = X - Y$. Show that $\rho(2U, 4V) = 0$.

2. A discrete random variable $X$ is Poisson with rate $\lambda > 0$, denoted by $Poi(\lambda)$, if its pmf is given by

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

(a) Prove that the mgf of a Poisson rv $X$ is

$$\phi_X(t) = \exp\{\lambda(e^t - 1)\}.$$  

(b) Use $\phi$ to compute the mean and variance of a rv that is $Poi(\lambda)$.

(c) If $X_1, \ldots, X_n$ are independent Poisson rvs with rates $\lambda_1, \ldots, \lambda_n$, respectively, show that $X_1 + \ldots + X_n$ is $Poi(\lambda_1 + \ldots + \lambda_n)$.

3. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with distributions function $F$. Define $F_n$ as follows: for any $a$

$$F_n(a) = \frac{\text{number of } X_i \text{ in } (-\infty, a]}{n}.$$  

Show that $\lim_{n \to \infty} P(|F_n(a) - F(a)| > \epsilon) = 0$ for any $\epsilon > 0$.

4. From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

(a) Based only on this information, give an upper bound to the probability that a student’s test score will exceed 85.

Suppose now that in addition the professor knows that the variance of a student’s test score is equal to 25.

(b) What can be said about the probability that a student will score between 65 and 85?

(c) How many students would have to take the examination so as to ensure, with probability at least .9, that the class average would be within 5 of 75?
5. Let $X$ and $Y$ be two independent random variables, where $X$ has a Bernoulli distribution with probability of success $p = 1/2$ and $Y$ has a Bernoulli distribution with probability of success $q = 1/4$. Determine $P(X + Y = k)$, for $k = 0, 1, 2$, and conclude that $X + Y$ does not have a binomial distribution.