Ordinary Differential Equations
Review Sheet

Zach Clawson

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1 Solving Techniques

1.1 Separation of Variables

1.1.1 Examples

Know how to do IVP problems like (and be able to analyze uniqueness of solutions):

- $y' = 2y$, $y(0) = 3$
- $u' = -u^{\frac{1}{3}}$, $u(0) = 1$
- $u' = u^{\frac{1}{3}}$, $u(0) = 0$

1.1.2 Theorems

Theorem about uniqueness:

Given $u' = \pm |u|^p$, $\begin{cases} p \geq 1 & \implies \text{unique solution} \\ p < 1 & \implies \text{check uniqueness} \end{cases}$

1.2 Integrating Factor

Solving $y' = \alpha \cdot y + \beta$, we can simply re-write it as $y' - \alpha \cdot y = \beta \implies (y \cdot e^{-\alpha \cdot t})' = \beta \cdot e^{-\alpha \cdot t}$. Integrate to find the solution.

2 Differential Inequalities

2.1 Definitions

- $y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$
- $y'_-(t) = \lim_{h \to 0^-} \frac{y(t+h) - y(t)}{h}$
- $y'_+(t) = \lim_{h \to 0^+} \frac{y(t+h) - y(t)}{h}$

2.2 Examples

Be able to solve problems like $p'_-(t) \leq \alpha \cdot p(t) + \beta(t)$.

2.3 Proofs to Know

Suppose $p : [a, b] \to \mathbb{R}$ and p is continuous.

Theorem 1 - If $p(a) < c$ and $p'_-(t) < 0$ whenever $p(t) = c$, then $p(t) < c \;\forall\; t \in [a, b)$.

Proof Let $p(a) < c$ and $p'_-(t) < 0$ whenever $p(t) = c$.

Suppose $p(t_1) = c$ for some $t_1 \in [a, b)$. Since p is continuous, there is a smallest $t_0 \in [a, b) \ni p(t_0) = c$.

Since $p(a) < c$, then $a < t_0 < b$. By our hypothesis we have $p'_-(t_0) < 0$. Also, since t_0 is the smallest t such that $p(t) = c$, then $p(t) < c \;\forall\; t \in [a, t_0)$.

Then for some "small" $h > 0$, $\frac{p(t_0-h)-p(t_0)}{-h} < 0 \implies p(t_0 - h) > c$.

But $t_0 - h < t_0$ and $p(t_0 - h) > c$. Then by the intermediate value theorem, since p is continuous, we have $\exists\; k \in (a, t_0 - h) \ni p(k) = c$. But $k < t_0$ so we have a contradiction. $\mathbb{QED}$

Theorem 2 - If $p(a) \leq c$ and $p'_+(t) < 0$ whenever $p(t) = c$, then $p(t) \leq c \;\forall\; t \in [a, b)$.

Proof Let $p(a) \leq c$ and $p'_+(t) < 0$ whenever $p(t) = c$.

Suppose $p(t_1) = c$ for some $t_1 \in [a, b)$. Since p is continuous, there must be a smallest $t_0 \in [a, b) \ni p(t_0) = c$.

Since $p(a) \leq c$, then $a \leq t_0 < b$. By our hypothesis we have $p'_+(t_0) < 0$. Also, since $t_0$is the smallest t such that $p(t) = c$, then $p(t) < c \;\forall\; t \in [a, t_0]$.

Assume for contradiction $\exists\; k \in [t_0, b) \ni p(k) > c$ and assume k_0 is the number "closest" to $t_0 \ni p(k) > c$.

Then for some "small" $h > 0$, $\frac{p(t_0+h)-p(t_0)}{h} < 0 \implies p(t_0 + h) < c$. But if we chose h small enough so that $t_0 + h < k_0$, then we arrive at a contradiction. $\mathbb{QED}$

3 Invariant Spaces

Let

$$(*)\;\; u' = f(u), \;\; u(0) = z, \;\; t \geq 0$$

3.1 Definition

We say a space Λ is **invariant** for $(*)$ iff $z \in \Lambda \implies u_z(t) \in \Lambda \;\forall\; t \geq 0$.

3.2 Theorem

$\Lambda = \left\{\vec{x} \in \mathbb{R}^N \,|\vec{a}_k \cdot (x, y) \leq \alpha_k, \, k = 1, 2, \ldots, m\right\}$is invariant for $(*)$
$\iff \vec{a}_k \cdot f(x, y) \leq 0$ when $\vec{a}_k \cdot (x, y) = \alpha_k$ for $k = 1, 2, \ldots m$.

3.3 Example

Let

$$(*) \;\; \begin{cases} x' = x - x^2 - xy & x(0) = x_0 > 0 \\ y' = y - y^2 - xy & y(0) = y_0 > 0 \end{cases}$$

Let $\Lambda = \left\{\vec{x} \in \mathbb{R}^2 \,|0 \le x \le \beta,\, 0 \le y \le \beta\right\} = \left\{ \vec{x} \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} (-1,\,0)\cdot(x,\,y) \le 0 \\ (0,-1)\cdot(x,\,y) \le 0 \\ (1,\,0)\cdot(x,\,y) \le \beta \\ (0,\,1)\cdot(x,\,y) \le \beta \end{array} \right\}$ for some $\beta \ge 1$. We want to show when $\vec{a}_k \cdot (x,\,y) = \alpha_k$ that $\vec{a}_k \cdot f(x,\,y) \le 0,\; k = 1,\,2,\,3,\,4.$

1. Let $k = 1$, then $x = 0$.
   $\vec{a}_1 \cdot f(0, y) = (-1,\,0) \cdot \big(0,\, y - y^2\big) = 0 \le 0$
2. Let $k = 2$, then $y = 0$.
   $\vec{a}_2 \cdot f(x, 0) = (0, -1) \cdot \big(x - x^2,\, 0\big) = 0 \le 0$
3. Let $k = 3$, then $x = \beta$.
   $\vec{a}_3 \cdot f(\beta,\, y) = (1,\,0) \cdot \big(\beta - \beta^2 - \beta y,\, y - y^2 - \beta y\big) = \beta - \beta^2 - \beta y \le \beta - \beta^2 = \beta(1-\beta) \le 0$ since $\beta \ge 1$.
4. Let $k = 4$, then $y = \beta$.
   $\vec{a}_4 \cdot f(x,\, \beta) = (0,\,1) \cdot \big(x - x^2 - \beta x,\, \beta - \beta^2 - \beta x\big) = \beta - \beta^2 - \beta y \le \beta - \beta^2 = \beta(1-\beta) \le 0$ since $\beta \ge 1$.

$\therefore$For $\beta \ge 1$, Λ is invariant for $(*)$.

4 Systems of Linear ODEs

Given

$$(\mathbf{H})\;\; u' = \mathbf{A}(t)u(t),\;\; u\,(t_0) = z,\;\; t \ge t_0$$

4.1 Matrix Solutions

We say $\mathbf{X}(t)$ is a matrix solution to $(\mathbf{H})$ iff

$$(\mathbf{MH})\;\; \mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)\;\forall\, t \textbf{ on the domain}$$

To get a $N \times N$ matrix $\mathbf{X}(t)$ we must find solutions to $(\mathbf{H})$, $\vec{x}_1(t),\,\ldots,\,\vec{x}_N(t)$ and write $\mathbf{X}(t) = (\vec{x}_1(t),\,\ldots,\,\vec{x}_N(t))$. Note that here, we can have $\vec{x}_i(t) = \vec{x}_j(t)$.

4.2 Fundamental Matrix Solutions

To get a FMS, you must first get a matrix solution which has N linearly independent columns, then write your FMS as:

$$\mathbf{T}_{\mathbf{A}}(t) = \mathbf{X}(t)\left[\mathbf{X}\,(t_0)\right]^{-1}$$

4.2.1 Properties of FMS

1. $\mathbf{T}_{\mathbf{A}}(0) = \mathbf{I}$
2. $\mathbf{T}_{\mathbf{A}}(s+t) = \mathbf{T}_{\mathbf{A}}(s) + \mathbf{T}_{\mathbf{A}}(t)$
3. $\left[\mathbf{T}_{\mathbf{A}}(t)\right]^{-1} = \mathbf{T}_{\mathbf{A}}(-t)$

4.2.2 Alternate Form of FMS

$\mathbf{T}_{\mathbf{A}}(t) = e^{t\mathbf{A}}$ can be shown using many methods. For simplicity we have $t_0 = 0$. This means

$$\mathbf{T}_{\mathbf{A}}(t) = e^{t\mathbf{A}} = \sum_{i=0}^{N} \frac{(t\mathbf{A})^i}{i!} = 1 + t\mathbf{A} + \frac{(t\mathbf{A})^2}{2!} + \cdots$$

4.2.3 Eigenvalues and Eigenvectors

We can use eigenvalues and eigenvectors to determine the form of $\mathbf{T_A}(t)$.

Eigenvalue - λ is an eigenvalue of $\mathbf{A} \iff \mathbf{A}\vec{v} = \lambda\vec{v}$ where $\vec{v} \neq 0$. **Eigenvector -** $\vec{v}$ is an eigenvector for $\lambda \iff \vec{v}$ satisfies the above equation for a given λ.

5 Phase Plots

Should review using eigenvalues and eigenvectors to determine phase plots of 2×2 systems.

6 Stability

Given

$$(*)\ u' = f\left(u,\, t\right),\ \ u\left(t_0\right) = z,\ \ t \geq t_0$$

and $w(t)$ is a given function for $t \geq t_0$, then we define the following:

- **Stability** - If we start "close" to $w(t)$, we stay close to that solution.

  $w(t)$ is **(S)** on $[t_0,\, \infty) \iff$

  $$\forall\, \epsilon > 0\ \exists\ \delta(t_0,\, \epsilon) \ni ||u(t_0) - w(t_0)|| < \delta$$
  $$\implies ||u(t) - w(t)|| < \epsilon\ \forall\ t \geq t_0$$

- **Uniform Stability**

  $w(t)$ is **(US)** on $[t_0,\, \infty) \iff$

  $$\forall\, \epsilon > 0\ \exists\ \delta(\epsilon) \ni ||u(t_1) - w(t_1)|| < \delta \text{ for any } t_1 \geq t_0$$
  $$\implies ||u(t) - w(t)|| < \epsilon\ \forall\ t \geq t_0$$

- **Asymptotic Stability** - If $w(t)$ is **(S)** and we start close to $w(t)$, then we go to $w(t)$.

  $w(t)$ is **(AS)** on $[t_0,\, \infty) \iff$

  $$w(t) \text{ is } \textbf{(S)} \text{ and } \exists\, \eta(t_0) > 0 \ni ||u(t_0) - w(t_0)|| < \eta \implies \lim_{t\to\infty} ||u(t) - w(t)||$$

- **Uniform Asymptotic Stability** - $w(t)$ is **(AS)** and $u(t)$ converges to $w(t)$ at a uniform rate (or faster for higher t_0)

  $w(t)$ is **(UAS)** on $[t_0,\, \infty) \iff$

  $$w(t) \text{ is } \textbf{(AS)} \text{ and } \exists\, \eta > 0 \ni \forall\ \epsilon > 0\ \exists\ T(\epsilon) \ni ||u(t_1) - w(t_1)|| < \eta$$
  $$\implies ||u(t) - w(t)|| < \epsilon \forall\ t \geq t_1 + T(\epsilon)$$

- **Exponential Stability** - $u(t)$ goes to $w(t)$ exponentially when we start "close enough"

  $w(t)$ is **(ES)** on $[t_0,\, \infty) \iff$

  $$\exists\, \alpha(t_0) > 0 \ni \forall\ \epsilon > 0\ \exists\ \delta(t_0,\, \epsilon) \ni ||u(t_0) - w(t_0)|| < \delta$$
  $$\implies ||u(t) - w(t)|| \leq e^{-\alpha\cdot(t-t_0)}$$

- **Uniform Exponential Stability** - $w(t)$ is **(ES)** and converges uniformly

  $w(t)$ is **(UES)** on $[t_0,\, \infty) \iff w(t)$ is **(ES)** $\forall\ t \geq t_0$ and α, δ are independent of t_0

7 Lyapunov Theory

7.1 Definition

Let $\Lambda \subset \mathbb{R}^N,\ V : \Lambda \to [0, \infty)$. If V is a Lyapunov function, then

1. V is positive-definite (with respect to $w \in \Lambda$)
   I.e. $V[w] = 0$ and $V[x] > 0$ if $x \in \Lambda,\ x \neq w$
2. V is Lipschits continuous on each bounded subset of Λ
   I.e. $\forall R > 0\ \exists\ L_R > 0 \ni |V[x] - V[y]| \leq L_R \cdot ||x - y||\ \ \forall\ x, y \in \Lambda,\ ||x||,\ ||y|| \leq R$

7.2 Property

Let $\mathcal{V}(\Lambda)$ contain all $V : \Lambda \to [0, \infty) \ni\ \forall\ R > 0,$

1. $\exists\ a \in CIP\,([0, \infty))$ (continuous, $\uparrow$(strictly), $a(0) = 0$, $a(r) > 0$) and $b > 0 \ni a(||x||) \leq V[x] \leq b \cdot ||x||$.
2. V is Locally Lipschitz

7.3 Derivative of V

We can take the derivative of V in two ways:

1. $\frac{d}{dt}V[(x,\, y)] = \frac{d}{dx}V[(x,\, y)] \cdot \frac{dx}{dt} + \frac{d}{dy}V[(x,\, y)] \cdot \frac{dy}{dt}$
2. $\frac{d}{dt}V[u(t)] = \vec{\nabla}V[u(t)] \cdot \frac{du}{dt} = \vec{\nabla}V[u(t)] \cdot f(t,\, u(t))$
3. $\frac{d^{\pm}}{dt}V\,[t_0,\, x] = D^{\pm}_{(*)}V\,[t_0,\, x] = \lim_{h \to 0^{\pm}} \frac{V[x + h \cdot f(t,x)] - V[x]}{h}$

7.4 Important Lyapunov Theorems

7.4.1 Lyapunov Stability Theorem (US)

Suppose $\exists\ V \in \mathcal{V}(\Lambda) \ni$

$$(\mathbf{1})\ \ D^{\pm}_{(*)}V[t,\, x] \leq 0\ \forall\ (t,\, x) \in [0,\, \infty) \times \Lambda,\ ||x|| \leq R$$

Then the zero solution to $(*)$ is **(US)**.

<u>Proof</u> Let $0 < \epsilon < R,\ t_0 \geq 0$ be given. Then we have $a(||x||) \leq V[x] \leq b \cdot ||x||\ \forall\ x \in \Lambda,\ ||x|| < R$

$\frac{d^{\pm}}{dt}V[u(t)] = D^{\pm}_{(*)}V[t, u(t)] \leq 0$ by $(\mathbf{1})$ $\qquad\qquad \implies V$ is non-increasing

So then we get:

$$(\mathbf{2})\ \ V[u(t)] \leq V\,[u\,(t_0)]\ \ \forall\ t \geq t_0 \ni ||u(t)|| \leq R$$

Choose $\delta = \delta(\epsilon) > 0 \ni b \cdot \delta < a(R)$ (i.e., $b \cdot \delta \in \text{Rng}(a)$) and $a^{-1}(b \cdot \delta) < \epsilon$. (Remember $||u\,(t_0)|| < \delta$)

So $a\,(||u(t)||) \leq V\,[u(t)] \leq V\,[u\,(t_0)] \leq b \cdot ||u\,(t_0)|| < b \cdot \delta \implies a\,(||u(t)||) < b \cdot \delta \implies ||u(t)|| < a^{-1}\,(b \cdot \delta) < \epsilon$

$\therefore$ The zero solution to $(*)$ is **(US)** .

QED

7.4.2 Lyapunov Asymptotic Stability Theroem (AS)

Suppose V, $W \in \mathcal{V}(\Lambda)$ and

$$\textbf{(3)}\;\; D^{\pm}_{(*)}V[t,\,x] < -W[x] \leq 0 \;\; \forall\, x \in \Lambda,\, ||x|| < R$$

Then the zero solution to $(*)$ is **(AS)**.

<u>Proof</u> $(*)$ is **(US)** by the previous theorem so we want to show **(AS)**. Assume $a(||x||) \leq V[x] \leq b \cdot ||x||$ and $\bar{a}(||x||) \leq W[x] \leq \bar{b} \cdot ||x|| \;\; \forall x \in \Lambda,\, ||x|| \leq R$.

Let η be $\ni$ if $||u\,(t_0)|| < \eta$ then $\bar{a}(||u(t)||) \leq \bar{a}(R) \;\forall\, t \geq t_0 \geq 0$. We claim the following:

$$\textbf{(4)}\;\; ||u\,(t_0)|| < \eta \implies \lim_{t \to \infty} V[u(t)] = 0$$

For contradiction, suppose $||u\,(t_0)|| < \eta$ but $\lim_{t\to\infty} V[u(t)] \neq 0$. Since $V[u(t)] \geq 0 \;\forall\, t \geq t_0 \geq 0$, this is equivalent to saying $\lim_{t\to\infty} V[u(t)] > 0$.

$$D^{\pm}_{(*)}V[t,\,u(t)] < -W[u(t)] \leq 0 \implies V[u(t)] \text{ is decreasing}$$

Thus we have $V[u(t)]$ is decreasing and $\lim_{t\to\infty} V[u(t)] > 0$ which means $\exists\, \alpha > 0 \ni V[u(t)] \geq \alpha \;\forall\, t \geq t_0 \geq 0$. Then we have $b \cdot ||u(t)|| \geq V[u(t)] \geq \alpha \implies ||u(t)|| \geq \frac{\alpha}{b} \;\forall\, t \geq t_0$.

Then $W[u(t)] \geq \bar{a}(||u(t)||) \geq \bar{a}\left(\frac{\alpha}{b}\right) \implies D^{\pm}_{(*)}V[u(t)] \leq -W[u(t)] \leq -\bar{a}\left(\frac{\alpha}{b}\right) \implies V[u(t)] - V\,[u\,(t_0)] \leq -\bar{a}\left(\frac{\alpha}{b}\right) \cdot (t - t_0)$; here we integrate from t_0 to t
$\implies V[u(t)] \leq V\,[u\,(t_0)] - \bar{a}\left(\frac{\alpha}{b}\right) \cdot (t - t_0) \to -\infty$ as $t \to \infty$
$\implies V[u(t)] \to -\infty$ which is a contradiction.

Thus, $\lim_{t\to\infty} V[u(t)] = 0$.

Now, $a(||u(t)||) \leq V[u(t)] \to 0$ as $t \to \infty \implies ||u(t)|| \leq a^{-1}\,(V\,[u\,(t)]) \to 0$ since $V[u(t)] \to 0$ and $a(0) = 0 \implies a^{-1}(0) = 0$ all as $t \to \infty$.

$$\therefore \lim_{t\to\infty} ||u(t)|| = 0 \implies \text{The zero solution to } \textbf{(*)} \text{ is } \textbf{(AS)}.$$

QED

7.5 Note About Positive-Definite Lyapunov Functions

If $V[(x, y)] = 2y^2$, V is not positive-definite since the only place where V should be 0 is at $(x,\, y) = (0,\, 0)$, but here V is 0 at a point such as $(1,\, 0)$.

7.6 Norms As Lyapunov Functions

Define $M_{\pm}[z,\, x] = \lim_{h\to 0^{\pm}} \frac{||z+h\cdot x||-||z||}{h}$ as the left and right directional derivatives of $V[x]$ at z in the direction of x. If $M_{\pm}[z,\, x]$ both exist and are equal, then $V[x]$ is differentiable at z.

7.6.1 Properties of $M_\pm[z,\, x]$

1. $M_-[z,\, x] \leq M_+[z,\, x]$
2. $M_\mp[z,\, x] = -M_\pm[z,\, x]$
3. $|M_\pm[z,\, x]| \leq ||x||$
4. $M_+[z,\, x+y] \leq M_+[z,\, x] + M_+[z,\, y]$ ("sublinear")
   and $M_-[z,\, x+y] \geq M_-[z,\, x] + M_-[z,\, y]$ ("superlinear")
5. $M_\pm[s \cdot z,\, r \cdot x] = r \cdot M_\pm[z,\, x],\; s,\, r > 0$ (linear with respect to x)
6. $M_\pm[z,\, \gamma \cdot z] = \text{Re}(\gamma) \cdot ||z|| \; \forall \; \gamma \in \mathbb{C}$
7. $|M_\pm[z,\, x] - M_\pm[z,\, y]| \leq ||x-y||$ (Lipschitz $\rightarrow$ use triangle inequality)
8. $M_\pm[z,\, x + \gamma \cdot z] = M_\pm[z,\, x] + \text{Re}(\gamma) \cdot ||z||$

7.6.2 Bounding $M_\pm[x,\, \mathbf{A}x]$

$\mu[\mathbf{A}]$ is the smallest α such that $M_\pm[x,\, \mathbf{A}x] \leq \alpha \cdot ||x||$. Note that if we can find one value of x such that $M_\pm[x,\, \mathbf{A}x] > 0$, then we immediately know $\mu[\mathbf{A}] > 0$.

7.6.3 Computing $M_\pm[z,\, x]$

If we're using $||\cdot||_1$ or $||\cdot||_\infty$ as our norm, then it is very hard to compute $M_\pm[z,\, x]$.

When we have a norm such as $||\cdot||_2$ though, it is slightly easier. We can write $M_\pm[z,\, x] = \frac{<z,x>}{||z||}$ where we define $< z, x >$ as the dot product and $||\cdot|| = ||\cdot||_2$. Note here that $||x||_2 = \sqrt{< x, x >}$. When showing what $M_\pm[z,\, x]$ is, use inner product properties (the dot product is one).

7.6.4 Important Theorems

Remember

$$(\mathbf{LH}) \;\; u' = \mathbf{A}u, \;\; u(0) = z, \;\; \mathbf{T_A}(t) = e^{t\mathbf{A}}$$

then we have the following two important theorems:

Theorem

1. $\left|\left|e^{t\mathbf{A}}\right|\right| \leq 1 \; \forall \; t \geq 0 \quad \Longleftrightarrow \quad \mu[\mathbf{A}] \leq 0$
2. $\left|\left|e^{t\mathbf{A}}\right|\right| \leq e^{-\alpha \cdot t} \; \forall \; t \geq 0 \quad \Longleftrightarrow \quad \mu[\mathbf{A}] \leq -\alpha$

8 Linearization of ODEs

Let $f = (f_i)_1^N$, where $f_i : \mathbb{R}^N \rightarrow \mathbb{R}$. We have $\vec{\nabla} f_i = \left(\frac{\partial f_i}{\partial x_1}, \, \ldots, \, \frac{\partial f_i}{\partial x_N}\right)$. Then we define the Jacobian of f is defined to be $\mathbf{D}f(\vec{z}) = \begin{pmatrix} \vec{\nabla} f_1 (\vec{z}) \\ \vdots \\ \vec{\nabla} f_N (\vec{z}) \end{pmatrix}$.

Let $\text{Re}(\lambda) < 0$ for $\lambda \in \sigma(\mathbf{A})$ and define

$$(\mathbf{PDE}) \;\; u' = \mathbf{A}u + g(t,\, u)$$

and let $\vec{z} \in \Lambda \ni f(\vec{z}) = \vec{0}$.

Theorem - (AS) by Linearization Suppose f is continuous, $f(\vec{z}) = \vec{0}$ and $\mathbf{D}f(\vec{z})$ exists (always does in C_1). If $\text{Re}(\lambda) < 0 \ \forall \ \lambda \in \sigma(\mathbf{D}f(\vec{z}))$, then the steady state solution $u(t) \equiv \vec{z}$ is **(ES)**.

Example Let $\alpha \in \mathbb{R}$ and $\begin{cases} x' = -x + \sin(\alpha \cdot y) \\ y' = x - y + x^2 \end{cases}$ with $(x, y) = (0, 0)$ as a critical point. Analyze the stability of the system with parameter α. Use linearization: $\mathbf{D}f(x, y) = \begin{pmatrix} -1 & \alpha\cos(\alpha \cdot y) \\ 1 + 2x & -1 \end{pmatrix} \Longrightarrow$ $\mathbf{D}f(0, 0) = \begin{pmatrix} -1 & \alpha \\ 1 & -1 \end{pmatrix}$ which has eigenvalues $\lambda = -1 \pm \sqrt{\alpha}$. The critical values for this eigenvalue are $\alpha = 0, 1$.

$$\begin{cases} \alpha < 0 & \Longrightarrow & \text{Re}(\lambda) < 0 \\ \alpha = 0 & \Longrightarrow & \text{Re}(\lambda) = -1 < 0 \\ 0 < \alpha < 1 & \Longrightarrow & \text{Re}(\lambda) < 0 \\ \alpha = 1 & \Longrightarrow & \text{one ev is } > 0 \\ \alpha > 1 & \Longrightarrow & \text{at least one ev is } > 0 \end{cases}$$

$\therefore$ We can conclude that the system is **(AS)** when $\alpha < 1$.