MA 515
Test 1 Study Guide

Metric Spaces

Metric space. (X, d) is a *metric space* if and only if $d : X \times X \to [0, \infty)$ is a function satisfying

1. Positive definite, $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
2. Symmetric, $d(x, y) = d(y, x)$ for all $x, y \in X$
3. Triangle inequality, $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Example metric spaces.

1. $X = l^\infty = \{\text{bounded real sequences}\} = \left\{x = \{x_i\}_{i \in \mathbb{N}} \mid x_i \in \mathbb{R} \text{ and } \sup_{i \in \mathbb{N}} |x_i| = M_x < \infty\right\}$ and $d(x, y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$
2. $X = B(A)$ where $A \subset \mathbb{R}$ and $B(A) = \{f : A \to \mathbb{R} \mid f \text{ is bounded}\}$ with $d(f, g) = \sup_{t \in A} |f(t) - g(t)|$.
3. $X = \mathcal{C}[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\} \subseteq B[a, b]$ (Why? Because continuous function on compact set is bounded.)
4. $X =$ any set and $d(x, y) = \begin{cases} 1 & , \quad x \neq y \\ 0 & , \quad x = y \end{cases}$.
5. (X, d) a metric space then (X, d') is also a metric space where $d'(x, y) = \frac{d(x,y)}{a + d(x,y)}$ with $a > 0$ fixed.
6. (X_0, d_0) is a metric space then $(\mathcal{S}, d)$ is a metric space where $\mathcal{S} = \left\{x = \{x_i\}_{i \in \mathbb{N}} : \mathbb{N} \to x_0 \mid x_i \in X_0 \ \forall\ i \in \mathbb{N}\right\}$ and $d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_0(x_k, y_k)}{a + d_0(x_k, y_k)}$ where $a > 0$ and note that $x, y \in \mathcal{S} \implies x, y$ are sequences in X_0. We can further define $d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_1(x_k, y_k)$ where d_1 is any general bounded metric.
7. $X = l^p = \left\{x \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} |x_i|^p < \infty\right\}$ with $p \geq 1$ fixed.

Hölder's inequality. For $p > 1$,

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1/q}$$

where p, q are conjugates of one another, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Minkowsky's inequality. For $p \geq 1$,

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

L^p spaces (Lesbesgue). Hölder's inequality becomes $\int |f \cdot g| du \leq \left(\int |f|^p du\right)^{1/p} \left(\int |g|^q du\right)^{1/q}$ where p, q are conjugates and Minkowsky's becomes $\left(\int |f + g|^p du\right)^{1/p} \leq \left(\int |f|^p du\right)^{1/p} + \left(\int |g|^p du\right)^{1/p}$.

Analysis Definitions

Open ball. $B(x,r) = \{y \in X \mid d(x,y) < r\}$

Closed ball. $\tilde{B}(x,r) = \{y \in X \mid d(x,y) \leq r\}$

Sphere. $S(x,r) = \{y \in X \mid d(x,y) = r\}$

Open set. $A \subseteq X$ is open $\iff \forall\, x \in A\ \exists\, \delta > 0 \ni B(x,\delta) \subseteq A \iff A = \mathring{A}$

Interior point. $x \in M \subseteq X$, x is an interior point of $M \iff \exists\, \delta > 0 \ni B(x,\delta) \subseteq M$

Interior set. $\mathring{A} = \{x \in A \mid x \text{ is an interior point of } A\}$

Accumulation point. x is an accumulation point of $M \subseteq X \iff \forall\, \epsilon > 0,\ (B(x,\epsilon)\backslash\{x\}) \cap M \neq \emptyset$.

Accumulation set. $\text{acc}(M) = \{x \in X \mid x \text{ is an accumulation point}\}$

Closure. $M \cup \text{acc}(M) = \bar{M}$

Topology. $(X,\mathcal{F}), \mathcal{F} \subseteq \mathcal{P}(X)$ is a topological space. $\mathcal{F}$ must satisfy

1. $\emptyset, X \in \mathcal{F}$
2. $\mathcal{F}$ closed under $\cup$
3. $\mathcal{F}$ closed under finite $\cap$

Convergent sequence. $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, x_n conergent to $x \in X \iff \{d(x_n,x)\}_{n\in\mathbb{N}} \to 0 \iff \forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \ni d(x_n,x) < \epsilon$ only if $n \geq N$

Continuous. $T : X \to Y$ ($(X,d),(Y,d)$ metric spaces) is continuous (at x) $\iff \forall\, \epsilon > 0,\ \exists\, \delta > 0 \ni d(a,x) < \delta \implies d(T(a),T(x)) < \epsilon$ (where $a \in X$) $\iff \forall\, \epsilon > 0,\ \exists\, \delta > 0 \ni \forall\, a \in B(x,\delta), T(a) \in B(T(x),\epsilon) \iff \forall\, \epsilon > 0,\ \exists\, \delta > 0 \ni T(B(x,\delta)) \subseteq B(T(x),\epsilon)$.

Bounded. $M \subseteq X$ is bounded $\iff \delta(M) = \text{diam}(M) = \sup_{x,y\in M} d(x,y) < \infty$

Separable Metric Space Examples

Dense set. $A \subseteq X$ is *dense* $\iff \forall\, x \in X,\ \forall\, \epsilon > 0, B(x,\epsilon) \cap A \neq \emptyset$

Separable. X is *separable* $\iff \exists\, A \subseteq X \ni A$ is dense and countable ($\bar{A} = X, |A| = \aleph_0$)

1. $\mathbb{Q} \subsetneq \mathbb{R}$ is dense and countable $\implies \mathbb{R}$ is separable.
2. $\mathbb{Q}^d \subsetneq \left(\mathbb{R}^d, \|\cdot\|_2\right)$ is dense and countable $\implies \mathbb{R}^d$ is separable
3. $(\mathbb{C}, |\cdot|)$ is analogous to $\left(\mathbb{R}^2, \|\cdot\|_2\right) \implies \mathbb{C}$ is separable metric space (e.g. $\{q_1 + iq_2 \mid q_i \in \mathbb{Q}\}$ is dense and countable)
4. l^∞ is not separable.

   Proof. Let $K = \{0,1\}^{\mathbb{N}} = \{\text{sequences of 0's and 1's only}\} \subsetneq l^\infty$ (as sequences of 0's and 1's must be bounded). Let $x, y \in K \ni x \neq y$. Then if $\epsilon = \frac{1}{3}$, $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$. Since there are uncountably many sequences of 0's and 1's then there also exist uncountably many balls about these sequences. Thus $\{B(x,\epsilon) \mid x \in K\}$ is uncountable. Let M be any dense set in l^∞. Then every ball of $\{B(x,\epsilon) \mid x \in K\}$ must contain an element in M. But these balls are non-intersecting so each one contains at least 1 distinct point in M. Thus there are uncountably many of these distinct points in M and therefore M must be uncountable. Therefore every dense set in M is uncountable and therefore l^∞ cannot be separable.

5. $l^p = \left\{x \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} |x_i|^p < \infty\right\}$ is separable. $A = \left\{q \in \mathbb{Q}^{\mathbb{N}} \mid q_i = 0 \text{ except for finitely many } q_i\text{'s}\right\}$ is dense and countable.

   Proof. A is countable. We must show that A is dense. We WTS that $\forall\ x \in l^p$, $\forall\ \epsilon > 0\ \exists\ q \in A \ni d(q, x) < \epsilon$ with $q \neq x$.

   Thus let $x \in l^p \implies \sum_{i=1}^{\infty} |x_i|^p < \infty \implies \exists\ N \in \mathbb{N} \ni \sum_{i=N+1}^{\infty} |x_i|^p < \epsilon_1$. Let

$$\begin{aligned} q &= (q_1, \ldots, q_N, 0, 0, \ldots) \\ x &= (x_1, \ldots, x_N, x_{N+1}, \ldots) \end{aligned}$$

   Then

$$d(x, q) = \left(\sum_{i=1}^{N} |x_i - q_i|^p + \sum_{i=N+1}^{\infty} |x_i|^p \right)^{1/p}$$

   We have that $\mathbb{Q}$ is dense so choose $q_i \in \mathbb{Q}$, $1 \le i \le N$ such that $|q_i - x_i| < \epsilon_2 \implies d(x, q) = \left(\sum_{i=1}^{N} |x_i - q_i|^p + \sum_{i=N+1}^{\infty} |x_i|^p\right)^{1/p} < (N \cdot \epsilon_2^p + \epsilon_1)^{1/p} < \epsilon$ for choice of $\epsilon_1 = \frac{\epsilon^p}{2}$ and $\epsilon_2 = \frac{\epsilon}{(2N)^{1/p}}$.

Completeness

Cauchy sequence. $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ is Cauchy $\iff d(x_n, x_m) \to 0$ as $n, m \uparrow \infty \iff \forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \ni d(x_n, x_m) < \epsilon$ only if $n, m \ge N$.

Complete. X is complete $\iff$ all Cauchy sequences in X converge in X

Banach space. Complete normed vector space (contains Hilbert spaces).

Hilbert space. Banach space but with norm induced by inner product.

Theorems.

1. $M \subseteq X$ is closed $\iff \{x_n\} \subseteq M$ and $x_n \to x \implies x \in M$

   Proof. "$\implies$" Let $M \subseteq X$ be closed. Let $x \in \bar{M}$. If $x \in M$ then $\{x_n\}_{n \in \mathbb{N}} \ni x_n \equiv x$ is a sequence in M that converges to $x \in M$. Now let $x \notin M$ (i.e. $x \in \partial M$). Then it must be an accumulation point and thus $B(x, 1/n)$ contains an $x_n \in M$ such that $x_n \neq x$. This is a sequence with $x_n \to x$ because $1/n \to 0$ as $n \uparrow \infty$.

   "$\Longleftarrow$" Suppose that $\{x_n\} \subseteq M$ such that $x_n \to x \implies x \in M$. If $x \in M$ then $x \in \bar{M}$. Now suppose that $x \notin M$ but $x \in \bar{M}$. But then we have that $B(x, \epsilon_n)$ contains an x_n different from x and thus x must be an accumulation point of M and therefore $x \in \bar{M}$.

2. $M \subseteq X$, X complete. M complete $\iff$ M closed.

   Proof. Let $M \subseteq X$ and X be complete.

   "$\implies$" Suppose M is complete. Then all Cauchy sequences in M converge to a point in M. $\{x_n\}$ Cauchy in $M \implies x_n \to x \in M$ and by (1) M is closed.

   "$\Longleftarrow$" Suppose M is closed. Then for all $\{x_n\} \subseteq M$ such that $x_n \to x \implies x \in M$. Let $\{x_n\} \subseteq M$ be Cauchy $\implies \{x_n\} \subsetneq X$ Cauchy and X complete $\implies x_n \to x \in X$ but then since M is closed, by definition $x \in M$.

3. $T : X \to Y$ is continuous $\iff \forall\ V \subseteq Y$ open, $T^{-1}(V) \subset\subseteq X$ is open.

   Proof. "$\implies$"Suppose T is continuous and $V \subseteq Y$ is open. If $T^{-1}(V) = \emptyset$ then we are done as $\emptyset$ is open. Assume $T^{-1}(V) \neq \emptyset$. Let $x_0 \in T^{-1}(V) \implies y_0 = T(x_0)$ for $y_0 \in V$. V open $\implies V \supseteq B(y_0, \epsilon) = N$. T continuous $\implies T^{-1}(V) \supseteq B(x_0, \delta) = N_0$ such that $T(B(x_0, \delta)) = B(y_0, \epsilon)$. Since $N \subseteq V, N_0 \subseteq T^{-1}(V)$ so $T^{-1}(V)$ open because $x_0 \in V$ was arbitrary.

   "$\Longleftarrow$"Suppose that for all open $V \subseteq Y$, $T^{-1}(V)$ is open in X. Therefore $\forall\ x_0 \in X$ and any ϵ-neighborhood N of $T(x_0)$, the inverse image N_0 of N is open since N open and N_0 contains x_0. Thus $N_0 \supseteq B(x_0, \delta)$ such that $T(B(x_0, \delta)) = B(T(x_0), \epsilon)$. Therefore T is continuous at x_0 and therefore T is continuous as x_0 was arbitrary.

4. T continuous at $x \iff x_n \to x \implies T(x_n) \to T(x)$.

   Proof. "$\implies$" Assume T is continuous at $x \implies \forall\ \epsilon > 0,\ \exists\ \delta > 0 \ni d(T(x), T(x_0)) < \epsilon$ if $d(x, x_0) < \delta$. Let $x_n \to x_0$. Then $\exists\ N \in \mathbb{N}$ such that $n \geq N$, $d(x_n, x_0) < \delta$. Therefore, for $n \geq N$, $d(T(x_n), T(x_0)) < \epsilon$ by continuity and therefore $T(x_n) \to T(x_0)$ by definition.

   "$\impliedby$" Assume $x_n \to x_0 \implies T(x_n) \to T(x_0)$. Suppose for contradiction T is not continuous. Then $\exists\ \epsilon > 0$ such that $\forall\ \delta > 0,\ \exists\ x \neq x_0$ such that $d(x, x_0) < \delta$ but $d(T(x), T(x_0)) \geq \epsilon$. Take $\delta = \frac{1}{n}$, then we have an $\{x_n\}$ such that $d(x_n, x_0) < \frac{1}{n}$ and $d(T(x_n), T(x_0)) \geq \epsilon$. Clearly $x_n \to x_0$ by this definition but $T(x_n) \not\to T(x_0)$. Contradiction.

Examples of Complete Metric Spaces

1. l^∞ complete

   Proof. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in l^∞. Then for any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$d(x_n, x_m) = \sup_{i\in\mathbb{N}} \left| x_i^{(n)} - x_i^{(m)} \right| < \frac{\epsilon}{2}$$

   A fortiori we thus know that $\left| x_i^{(n)} - x_i^{(m)} \right| < \frac{\epsilon}{2}$ for every fixed i. Thus the sequence $\left(x_i^{(1)}, x_i^{(2)}, \ldots \right)$ is a Cauchy sequence in $\mathbb{R}$ and therefore it converges to, say, x_i. Therefore we define $x = (x_1, x_2, \ldots)$ to be the sequence of these limit points in i. First, we have that x is in l^∞ since for $x_N = \left(x_i^{(N)} \right)_{i\in\mathbb{N}}$ there is a number such that $\left| x_i^{(N)} \right| \leq K_N$. By the triangle inequality

$$|x_i| \leq \left| x_i - x_i^{(N)} \right| + \left| x_i^{(N)} \right| < \frac{\epsilon}{2} + K_N$$

   and the RHS does not depend on i so thus this must be true for all $i \in \mathbb{N}$. Thus $x \in l^\infty$. Now, since $\left| x_i^{(n)} - x_i^{(m)} \right| < \frac{\epsilon}{2}$ then letting $m \uparrow \infty$ we have that $\left| x_i^{(n)} - x_i \right| < \frac{\epsilon}{2}$ and therefore $d(x_n, x) = \sup_i \left| x_i^{(n)} - x_i \right| \leq \frac{\epsilon}{2} < \epsilon$ and therefore $x_n \to x \in l^\infty$.

2. $l^p, 1 \leq p < \infty$ complete

   Proof. Remember $d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq l^p$ be Cauchy. Then

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } d\left(x_n, x_m\right)^p = \sum_{i=1}^{\infty} \left| x_i^{(n)} - x_i^{(m)} \right| < \left(\frac{\epsilon}{2} \right)^p \text{ only if } n, m \geq N$$

   and thus $\left| x_i^{(n)} - x_i^{(m)} \right| < \left(\frac{\epsilon}{2} \right)^p$ for $n, m \geq N$ and for all $i \in \mathbb{N}$. Thus for a fixed $i \in \mathbb{N}$, $\left\{ x_i^{(n)} \right\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R} \implies x_i^{(n)} \to x_i$ as $n \uparrow \infty$. But the for $\epsilon > 0$, choosing the same N as before,

$$\sum_{i=1}^{r} \left| x_i^{(n)} - x_i^{(m)} \right|^p < \left(\frac{\epsilon}{2} \right)^p \implies \lim_{m\to\infty} \sum_{i=1}^{r} \left| x_i^{(n)} - x_i^{(m)} \right|^p < \lim_{m\to\infty} \left(\frac{\epsilon}{2} \right)^p$$
$$\implies \sum_{i=1}^{r} \left| x_i^{(n)} - x_i \right|^p \leq \left(\frac{\epsilon}{2} \right)^p$$

   Note that this sum is now an increasing sequence (with respect to r) and bounded above. Thus it converges to

$$\sum_{i=1}^{\infty} \left| x_i^{(n)} - x_i \right|^p \leq \left(\frac{\epsilon}{2} \right)^p \implies \left(\sum_{i=1}^{\infty} \left| x_i^{(n)} - x_i \right|^p \right)^{1/p} \leq \frac{\epsilon}{2} < \epsilon$$

and therefore $d(x_n, x) < \epsilon$ and thus $x_n \to x$. All that remains is to show that $x \in l^p$. By the Minkowsky Inequality we may write $x = x_m + (x - x_m) \in l^p$, $m \geq N$, and thus

$$\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty} \left|x_i^{(m)} + \left(x_i - x_i^{(m)}\right)\right|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} \left|x_i^{(m)}\right|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} \left|x_i - x_i^{(m)}\right|^p\right)^{1/p}$$

and the first term is $< \infty$ since $x_m \in l^p$ and the second one is $\leq \frac{\epsilon}{2}$ and thus the sum is finite, so $x \in l^p$.

3. $\mathcal{C}[a, b]$ complete

   Comment. $\mathcal{C}[a, b]$ is complete with respect to the norm $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$ but is not complete with respect to $d(f, g) = \int_a^b |f(t) - g(t)| dt$ (induced by $L^1[a, b]$ due to $\mathcal{C}[a, b] \subseteq L^1[a, b] = \{f : [a, b] \to R \mid f \text{ is integrable, i.e. } d(f, g) < \infty\}$).

   Comment. A counter-example to the completeness of $\mathcal{C}[a, b]$ under the L^1 norm is by looking at $\mathbb{P}[a, b] \subsetneq \mathcal{C}[a, b]$, the set of polynomials on $[a, b]$ not complete. Counter-example is $f_n(x) = x^n$ on $[0, 1]$. $f_n(x) \to f(x) = \begin{cases} 0 & , \quad x \in [0, 1) \\ 1 & , \quad x = 1 \end{cases} \notin \mathbb{P}[a, b]$ (also not in $\mathcal{C}[a, b]$).

   Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}[a, b] \implies \forall\ \epsilon > 0\ \exists\ N \in \mathbb{N}$ such that $d(f_n, f_m) < \epsilon$ if $n, m \geq N \implies \sup_{t \in [a,b]} |f_n(t) - f_m(t)| < \epsilon \implies |f_n(t) - f_m(t)| < \epsilon\ \forall\ t \in [a, b], n, m \geq N \implies$ for fixed t_0, $\{f_n(t_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R} \implies f_n(t_0) \to f(t_0)$ as $n \uparrow \infty$. Therefore we have shown pointwise convergence of $f_n(t) \to f(t)$. We must show that $f \in \mathcal{C}[a, b]$. Note that since $\{f_n\}$ is Cauchy, then $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N}$ such that $\sup_{t \in [a,b]} |f_n(t) - f_m(t)| < \frac{\epsilon}{2}$ when $n, m \geq N$. Letting $m \uparrow \infty \implies |f_n(t) - f(t)| \leq \frac{\epsilon}{2} < \epsilon$ since $f_m \to f$. Therefore $f_n \to f$ uniformly so $f \in \mathcal{C}[a, b]$.

4. $\mathbb{Q}$ is not complete.

   Proof. Note that $\mathbb{Q} \subsetneq \mathbb{R}$ and $\mathbb{R}$ is complete. Thus it suffices to show $\mathbb{Q}$ is not closed in order to show $\mathbb{Q}$ is not complete. Note that $\pi \in \text{acc}(\mathbb{Q})$ since every ball about π contains a rational number. But $\pi \notin \mathbb{Q}$ and therefore we can construct a sequence in $\mathbb{Q}$ based on these balls that converge to π. Therefore $x_n \to \pi$ but $\pi \notin \mathbb{Q}$ and therefore $\mathbb{Q}$ is not closed and therefore not complete.

5. $c = \{\text{all convergent sequences}\}$ is complete

   Proof. Note that $c \subsetneq l^\infty$ and therefore c is complete $\iff$ c is closed. Also note that the metric on c is induced by l^∞, i.e. $d(x, y) = \sup_i |x_i - y_i|$. We let $x \in \bar{c}$, the closure of c. We want to show that $x \in c$. By definition of $\bar{c}$, there exists $x_n \to x$ where $\{x_n\} \subseteq c$. Thus for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\left|x_i^{(n)} - x_i\right| \leq \sup_i \left|x_i^{(n)} - x_i\right| = d(x_n, x) < \frac{\epsilon}{3}$$

   Note also that $x_n = \left\{x_i^{(n)}\right\}_{i \in \mathbb{N}}$ is itself a sequence in c that converges to x_i and thus it is Cauchy. Therefore there exists an N_1 such that for $n \geq N_1$,

$$\left|x_i^{(n)} - x_j^{(n)}\right| < \frac{\epsilon}{3}$$

   and therefore using the triangle inequality

$$\begin{aligned} |x_i - x_j| &\leq \left|x_i - x_i^{(n)}\right| + \left|x_i^{(n)} - x_j^{(n)}\right| + \left|x_j^{(n)} - x_j\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

   and therefore x is a Cauchy sequence of real numbers and therefore it converges. Therefore, by definition of c, $x \in c$ and therefore $\bar{c} \subseteq c$ and a priori we knew $c \subseteq \bar{c}$ so therefore $c = \bar{c}$ and c is closed.

Completion of Metric Spaces

Isometry. A map $T : X \to Y$ with respective metrics d_X and d_Y is an *isometry* if and only if it satisfies $d_X(a, b) = d_Y(T(a), T(b))$.

Isometric. Two metric spaces, X and Y, are said to be isometric if there exists a bijective isometry from X to Y.

Normed Vector Spaces / Banach Spaces

Vector space. $X = (X, +, \cdot)$ (space, addition of elements in space, scalar multiplication) over K (scalar space, a field, usually $\mathbb{R}$ or $\mathbb{C}$) is a *vector space* if and only if for $x, y, z \in X$ and $\alpha, \beta \in K$,

1. Closed under $+ : X \times X \to X$ with $(x, y) \mapsto x + y$
2. Closed under $\cdot : K \times X \to X$ with $(\alpha, x) \mapsto \alpha \cdot x$
3. $(x + y) + z = x + (y + z)$
4. There exists $0 \in X$ such that $x + 0 = 0 + x = x$
5. $x + y = y + x$
6. There exists $-x \in X$ such that $x + (-x) = (-x) + x = 0$
7. $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
8. There exists $1 \in K$ such that $1 \cdot x = x \cdot 1 = x$
9. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
10. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

Subspace. If X is a vector space, then $Y \subseteq X$ is a *subspace* $\iff$ Y is a vector space $\iff$ Y is closed under $+$ and $\cdot$.

Linear combination. If $x_i \in X$ and $\alpha_i \in K$ for $i = 1, 2, \ldots, n$ then $\sum_{i=1}^{n} \alpha_i \cdot x_i$ is a linear combination of elements in X.

Linear independence. $x_1, x_2, \ldots, x_n \in X$ are linearly independent $\iff$ $\sum_{i=1}^{n} \alpha_i \cdot x_i = 0 \implies \alpha_i = 0$ for all $i = 1, 2, \ldots, n$.

Span. For $M \subseteq X$, $\text{span}(M) = \{\text{all linear combinations of elements of } M\}$ is a subspace. M spans $X \iff \text{span}(M) = X$.

Basis. $B \subseteq X$ is a basis $\iff$ B is linearly independent and $\text{span}(B) = X$.

Dimension. If B is a basis for X, then $\dim X = |B|$ (cardinality of B)

Norm. A *norm* on a vector space X, $\|\cdot\| : X \to [0, \infty)$ defined by $x \mapsto \|x\|$ over $K = \mathbb{R}$ or $\mathbb{C}$ satisfies (for $x, y \in X$ and $\alpha \in K$)

1. Positive-definiteness: $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
2. Scalar multiplication: $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$
3. Triangle-inequality (sub-additivity): $\|x + y\| \leq \|x\| + \|y\|$

Semi-norm. $p : X \to [0, \infty)$ is a semi-norm $\iff$ it satisfies properties 2 and 3 above, but not necessarily 1 (i.e. some $x \in X$ that is not 0 could have $p(x) = 0$).

Quotient space. For X a vector space and N a subspace, X/N is a vector space.

Lesbesgue integral. $f \mapsto \|f\|$ is defined by $\|f\| = \int_a^b |f(x)|dx$. Note that positive-definiteness is not satisfied for any general f and thus this is a semi-norm. If f were continuous, then this would be a norm. Therefore we define X/N to be a normed vector space when $N = \ker X = \{g \in X \mid \|g\| = 0\}$.

Banach space. Complete normed vector space.

Hamel basis. A basis $\{e_\alpha\}_{\alpha \in I}$ is a Hamel basis $\iff \forall\, x \in X\ \exists!\ \{\alpha_n\} \subseteq K$ such that $x = \sum_{i=1}^{p} \alpha_i \cdot e_i$.

Schauder basis. Basis for a normed vector space X is $\{e_i\}_{i \in I}$ is a Schauder basis $\iff \forall\, x \in X\ \exists\, \{\alpha_i\}_{1 \leq i \leq \infty} \subseteq K$ such that $x = \sum_{i=1}^{\infty} \alpha_i \cdot e_i$.

Theorem (Banach). If X is a normed vector space with Schauder basis, then X is separable.

Proof. We WTS $\forall\, x \in X,\ \forall\, \epsilon > 0,\ \exists\, a \in M$ such that $d(a, x) < \epsilon$ for some $M \subseteq X$. I.e. we want to show there exists some dense subset of X and then show it is countable.

Let $x \in X$ and $\epsilon > 0$. Let $M = \cup_{n=1}^{\infty} A_n$ where $A_n = \{\sum_{i=1}^{n} q_i \cdot e_i \mid q_i \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is a dense countable subset of K, then a finite linear combination of elements in $\mathbb{Q}$ will be and then a countable union of countable sets is also countable. Therefore M is countable.

By the definition of the Schauder basis, $\exists\, N \in \mathbb{N}$ such that $\|x - \sum_{i=1}^{n} \alpha_i e_i\| < \frac{\epsilon}{2}$ if $n \geq N, \{\alpha_i\} \subseteq K$. And further, $\mathbb{Q}$ is a dense subset of $K \implies \forall\, \alpha_i\ \exists\, q_i$ such that $|\alpha_i - q_i| < \frac{\epsilon}{2b}$ where $b = \sum_{i=1}^{n} \|e_i\|$. Let $a = \sum_{i=1}^{n} q_i e_i \in M$.

$$\begin{aligned}
\|x - a\| &= \left\| x - \sum_{i=1}^{n} q_i e_i \right\| \\
&\leq \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| + \left\| \sum_{i=1}^{n} (\alpha_i - q_i) e_i \right\| \\
&< \frac{\epsilon}{2} + \sum_{i=1}^{n} |\alpha_i - q_i| \|e_i\| \\
&< \frac{\epsilon}{2} + \sum_{i=1}^{n} \frac{\epsilon}{2b} \cdot \|e_i\| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore M is dense and countable $\implies X$ is separable.

Bolzano-Weierstraus Theorem. Every bounded sequence has a convergent subsequence.

Observation of boundedness of vectors in X. If X is a normed vector space with Hamel basis given by $\{e_i\}_{1 \leq i \leq n}$ (linearly independent), then $\exists\, c, M \in K$ such that

$$c \cdot \sum_{i=1}^{n} |\alpha_i| \leq \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| \leq M \cdot \sum_{i=1}^{n} |\alpha_i|$$

Proof. Note that if we choose $M = \max_{1 \leq i \leq n} \|e_i\|$ then the $\leq$ part is trivial by the triangle inequality. Note that if $c \cdot \sum_{i=1}^{n} |\alpha_i| \leq \| \sum_{i=1}^{n} \alpha_i e_i \|$, then $c \leq \frac{\|\sum_{i=1}^{n} \alpha_i e_i\|}{\sum_{i=1}^{n} |\alpha_i|} = \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\sum_{i=1}^{n} |\alpha_i|} e_i \right\|$ and taking $B_i = \frac{\alpha_i}{\sum_{k=1}^{n} |\alpha_i|}$ (note that $\sum_{i=1}^{n} |\beta_i| = 1$) then we want to show that $\|\sum_{i=1}^{n} \beta_i e_i\| \geq c > 0$ where $\sum_{i=1}^{n} |\beta_i| = 1$. Let $M = \{x = (x_1, \ldots, x_n) \in K^n \mid \sum_{i=1}^{n} |x_i| = 1\}$.

For contradiction assume that $\exists\, \{\beta_k\}_{k \in \mathbb{N}} \subseteq K$ such that $\left\| \sum_{i=1}^{n} \beta_i^{(k)} e_i \right\| \to 0$ with $\beta_k = (\beta_1^{(k)}, \ldots, \beta_n^{(k)})$ satisfying $\sum_{i=1}^{n} |\beta_i^{(k)}| = 1$ for $k = 1, 2, \ldots$.

M is a bounded set in K $(\mathbb{R}^n, \mathbb{C}^n)$. Thus β_k is bounded and $\beta_k = (\beta_1^{(k)}, \ldots, \beta_n^{(k)}) \in M$. Bolzano Weierstraus Theorem says that there exists $\beta^{(k_r)}$ such that $\beta^{(k_r)} \to \gamma$ and since $\beta^{(k_r)} \in M$ and M is closed, then $\gamma \in M$. Thus $\sum_{i=1}^n |\gamma_i| = 1$. But $\sum_{i=1}^n \beta_i e_i \to \sum_{i=1}^n \gamma_i e_i$ and $\sum_{i=1}^n \beta_i e_i \to 0$ by our assumption. Thus $\sum_{i=1}^n \gamma_i e_i = 0 \implies \gamma_i = 0$ for all $i = 1, 2, \ldots, n$ since e_i's are linearly independent. But then $\gamma \notin M$ is our contradiction, as we showed it was.

Quotient Spaces

Let X be a normed vector space with scalar field K. Let N be a subspace of X. Then

$$X/N = \{x + N \mid x \in X\} \text{ is called a quotient space}$$

Define $\pi : X \to X/N$ by $\pi(x) = x + N$ and further define

$$\begin{aligned} \pi(x) + \pi(y) &= \pi(x + y) \\ \alpha \cdot \pi(x) &= \pi(\alpha \cdot x) \end{aligned}$$

Note that $\pi(x) = \pi(x') \implies \pi(x) - \pi(x') = 0 \implies \pi(x - x') = 0 \implies x - x' \in 0 + N \implies x - x' \in N$.

Define the equivalence relation $x \sim y \iff x - y \in N$. Thus $X/N = X/\sim$ and $\pi(x) = [x]$.

Suppose X is a vector space with a semi-norm $p(x)$. We want to show $(X/N, \|\cdot\|)$ is a normed vector space. Define $\|\pi(x)\| = p(x)$ and let $N = p^{-1}(\{0\})$. This is a normed vector space.

Theorem. X is a normed vector space $\implies$ X/N is a normed vector space $\iff$ $N \subsetneq X$ is a closed subspace.

Proof. All that must be shown is that the norm defined by $\|\pi(x)\| = d(x, N) = \inf_{y \in N} \|x - y\|$ is a norm (where the second norm is a norm in X).

Theorem. X is a Banach space $\implies$ X/N is a Banach space.

Proof. Must show that X/N is complete.