Section 3.5. Series Related to Orthonormal Sequences.

**Theorem (Convergence).** Let \((e_k)\) be an orthonormal sequence in a Hilbert space \(H\). Then:

(a) \[\sum_{k=1}^{\infty} \alpha_k e_k\] converges (in the norm on \(H\)) \(\iff\) \[\sum_{k=1}^{\infty} |\alpha_k|^2\] converges.

(b) \[x = \sum_{k=1}^{\infty} \alpha_k e_k\] converges \(\implies\) \[\alpha_k = |\langle x, e_k \rangle|\].

(c) \(x \in H, x = \sum_{k=1}^{\infty} \alpha_k e_k\) converges with \(\alpha_k = \langle x, e_k \rangle\).

**Proof.** (a) Let \(s_n = \sum_{i=1}^{n} \alpha_i e_i\) and note

\[
\|s_m - s_n\|^2 = \left\| \sum_{i=n+1}^{m} \alpha_i e_i \right\|^2 = \sum_{i=n+1}^{m} \|\alpha_i e_i\|^2 \quad \text{(Pythagorean theorem)}
\]

if we take \(t_k = \sum_{i=1}^{k} |\alpha_i|^2\). Thus if one converges then the other must.

(b) \[x = \sum_{i=1}^{\infty} \alpha_i e_i\] exists \(\iff\) \(s_n = \sum_{i=1}^{n} \alpha_i e_i \rightarrow x\). Note

\[
\langle x, e_i \rangle = \lim_{n \to \infty} \langle s_n, e_i \rangle = \lim_{n \to \infty, n > i} \langle s_n, e_i \rangle = \lim_{n \to \infty, n > i} \left\langle \sum_{j=1}^{n} \alpha_j e_j, e_i \right\rangle
\]

which is 0 if \(n \leq i\) so assume that \(n > i\) \(\implies\) every term is 0 except for \(i^{th}\) one = \(\alpha_i\). Thus \(\alpha_i = \langle x, e_i \rangle\) for all \(i \in \mathbb{N}\).

(c) By (a) we have that \(\sum_{i=1}^{\infty} |\langle \alpha, e_i \rangle|^2\) exists by Bessel Inequality.

Q.E.D.

**Lemma (Fourier coefficients).** Any \(x \in X\) inner product space can have at most countably many nonzero Fourier coefficients \(\langle x, e_k \rangle\) with respect to an orthonormal family \((e_k), k \in I, \text{ in } X\).

**Proof.** Let \(x \in H\) and write \(x = \sum_{\alpha \in J}(x, e_\alpha)e_\alpha\) for all \(x \in X\) which is an uncountable sum. But if we can show that there are a countable number of non-zero Fourier coefficients. Define the set for fixed \(x \in H, J_x = \{\alpha \in I \mid \langle x, e_\alpha \rangle \neq 0\}\subseteq I\). We may then write

\[
x = \sum_{\alpha \in I}(x, e_\alpha)e_\alpha = \sum_{\alpha \in J_x}(x, e_\alpha)e_\alpha
\]
We want to show that the set $J_x$ is countable. Define

$$J_k = \left\{ \alpha \in I \mid \langle x, e_\alpha \rangle > \frac{1}{k} \right\}$$

noting $J_k \subseteq J_{k+1}$ and defining $J = \bigcup_{k=1}^{\infty} J_k = \lim_{k \to \infty} J_k$. We want to show that each $J_k$ is countable in order to show that $J$ is countable (as we would obtain a countable union of countable sets). Choose $M \subseteq J_k$ such that $M = \{\alpha_1, \ldots, \alpha_m\} \subseteq J_k$ is a finite set. Then since $\langle x, e_\alpha \rangle > \frac{1}{k}$ we then have

$$m \cdot \frac{1}{k^2} < \sum_{i=1}^{m} |\langle x, e_\alpha \rangle|^2 \leq \|x\|^2 < \infty$$

and noting that the LHS $\uparrow \infty$ as $m \uparrow \infty$ gives a contradiction and thus $m$ must be fixed a priori and thus each $J_k$ must be finite, showing the countability of $J_x$.

Q.E.D.

Section 3.6. Total Orthonormal Sets and Sequences.

Total orthonormal set. A total set in a normed space $X$ is a subset $M \subseteq X$ whose span is dense in $X$. Accordingly, an orthonormal set in an inner product space $X$ which is total in $X$ is called a total orthonormal set in $X$. That is, $M$ is total in $X$ $\iff$ $\text{span}M = X$.

Theorem (totality). Let $M$ be a subset of an inner product space $X$. Then:

(a) If $M$ is total in $X$, then there does not exist a nonzero $x \in X$ which is orthogonal to every element of $M$; that is, $x \perp M \implies x = 0$.

(b) If $X$ Hilbert, then $x \perp M \implies x = 0$ shows $M$ total in $X$.

Facts. (a) $M$ is total in Hilbert $H$ $\iff$ $M^\perp = \{0\}$

(b) $M$ total in $H$ $\iff$ $\overline{\text{span}M} = H$

(c) $M$ total $\iff$ Parseval’s equality holds, i.e. $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2$

Theorem (Separable Hilbert spaces). Let $H$ be a Hilbert space. Then:

(a) If $H$ separable, every orthonormal set in $H$ is countable.

(b) If $H$ contains an orthonormal sequence which is total in $H$, then $H$ is separable.

Section 3.7. Legendre, Hermite and Laguerre Polynomials.

Legendre polynomials. Can represent them in many ways:

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} [(t^2 - 1)^n]$$

$$= \sum_{j=0}^{N} (-1)^j \frac{(2n - 2j)!}{2^j j!(n-j)!(n-2j)!} t^{n-2j}, \quad N = \frac{n}{2} \text{ or } \frac{n-1}{2} \text{ if even/odd}$$
First few polynomials given by:

\[
\begin{align*}
P_0(t) &= 1 \\
P_1(t) &= t \\
P_2(t) &= \frac{1}{2}(3t^2 - 1) \\
P_3(t) &= \frac{1}{2}(5t^3 - 3t) \\
P_4(t) &= \frac{1}{8}(35t^4 - 30t^2 + 3) \\
&\vdots
\end{align*}
\]

And applying G-S process we can arrive at

\[c_n = \sqrt{\frac{2n + 1}{2}} P_n(t)\]

Section 3.8. Representation of Functionals on Hilbert Spaces.

**Riesz Lemma.** Y is a closed subspace of normed vector space X \(\implies\) \(\forall \theta \in (0, 1) \exists x \in S_X(0, 1)\) such that \(d(x, Y) > \theta\).

**Riesz’s Theorem (RR Thm baby).** Every bounded linear functional \(f\) on a Hilbert space \(H\) can be represented in terms of the inner product, namely, \(f(x) = \langle x, z \rangle\) where \(z\) depends on \(f\), is uniquely determined by \(f\) and has norm \(\|f\|_H = \|f\|_{op}\).

**Proof.** See that for \(f \in H'\) we have \(f(f(x) \cdot a - f(a) \cdot x) = 0\) for all \(a, x \in H\) trivially. Then \(f(x)a - f(a)x \in N = \ker f\). \(N\) is a closed subspace of \(H\) and thus \(H = N \oplus N^\perp\). If \(N^\perp = \{0\}\) then \(H = N = \ker f \implies f \equiv 0\) so choose \(z = 0\) for the inner product.

If \(N^\perp \supseteq \{0\}\) then \(\exists a \in N\) with \(a \neq 0\) such that \(\langle f(x)a - f(a)x, \_a \rangle = 0 \implies f(x)||a||^2 = f(a)||a, a \rangle \implies f(x) = \frac{f(a)}{||a||^2} ||a, a \rangle \implies f(x) = \frac{f(a)}{||a||^2} (x, a) = \langle x, \frac{f(a)}{||a||^2} \cdot a \rangle \)

Q.E.D.

**Definition (Sesquilinear form).** Let \(X\) and \(Y\) be vector spaces over the same field \(K\) (\(\mathbb{R}\) or \(\mathbb{C}\)). Then a **sesquilinear form** (or sesquilinear functional) \(h\) on \(X \times Y\) is a mapping \(h : X \times Y \to K\) such that for all \(x, x_1, x_2 \in X\) and \(y, y_1, y_2 \in Y\) we have

\[
\begin{align*}
h(x_1 + x_2, y) &= h(x_1, y) + h(x_2, y) \\
h(x, y_1 + y_2) &= h(x, y_1) + h(x, y_2) \\
h(\alpha x, y) &= \alpha h(x, y) \\
h(x, \beta y) &= \beta h(x, y)
\end{align*}
\]

Note if \(K = \mathbb{R}\) then the last condition simply gives this is a **bilinear form**.

**Norm on \(h\).** \(h\) is bounded if \(|h(x, y)| \leq c\|x\|\|y\|\) for some \(c \in [0, \infty)\). The norm is given by

\[
\|h\| = \sup_{x \in X - \{0\}, y \in Y - \{0\}} \frac{|h(x, y)|}{\|x\|\|y\|} = \sup_{\|x\|=1,\|y\|=1} |h(x, y)|
\]

3
**Theorem (Riesz representation adult).** Let $H_1, H_2$ be Hilbert spaces and $h : H_1 \times H_2 \to \mathbb{K}$ a bounded sesquilinear form. The $h$ has representation $h(x, y) = \langle Sx, y \rangle$ where $S : H_1 \to H_2$ is a bounded linear operator. $S$ is uniquely determined by $h$ and has norm $\|S\| = \|h\|$.  

**Proof.** Fix $x \in H_1$ and let $f_x : H_2 \to \mathbb{K}$ defined by $f_x(y) = h(x, y)$ which is clearly bounded and linear. Bounded because $\|f\|_{op} \leq \|h\|_{sesq} \|x\|_H$. Thus by RR Theorem (baby) we have 

$$\exists! \ z_x \in H_2 \text{ s.t. } f_x(\cdot) = \langle \cdot, z_x \rangle_{H_2}$$

but we have that $f_x(\cdot) = \overline{h(x, \cdot)} \implies h(x, \cdot) = \langle \cdot, z_x \rangle_{H_2} \implies h(x, \cdot) = \langle z_x, \cdot \rangle_{H_2}$. Thus for any choice of $x$ we may form this relationship between $h$ and the inner product with choice of $z_x$.

Define $S : H_1 \to H_2$ by $Sx = z_x$. By construction we trivially have that $h(x, y) = \langle z_x, y \rangle = \langle Sx, y \rangle$ for all $y \in H_2$ for fixed $x \in H_1$. Linearity is easy to show. Must show bounded operator and norm-preserving:

"Bounded." WTS $\|Sx\|_{H_2} \leq c \cdot \|x\|_{H_1}$ for some $c \in [0, \infty)$. Note 

$$|\langle Sx, y \rangle| = |h(x, y)| \leq \|h\|_{sesq} \|x\|_{H_1} \|y\|_{H_2} \quad \forall \ y \in H_2$$

and choosing $y = Sx$ we thus have 

$$\|Sx\|^2 \leq \|h\|_s \|x\| \cdot \|Sx\| \implies \|Sx\| \leq \|h\|_s \|x\| \quad \text{(if } \|Sx\| = 0 \text{ then trivial)}$$

and thus $\|S\|_{op} \leq \|h\|_s$.

"Norm preserving." From RR Theorem (baby) we have $\|f_x\|_{op} = \|z_x\| = \|Sx\|$ and since $\|f_x\|_{op} = \sup_{\|y\|_{H_2} = 1} |f_x(y)| = \sup_{\|y\| = 1} |h(x, y)|$ we thus have 

$$\|Sx\| = \sup_{\|y\| = 1} |h(x, y)|$$

and then taking sup over $x \in H_1$ with $\|x\| = 1$ we thus have 

$$\sup_{\|x\| = 1} \|Sx\| = \sup_{\|x\| = 1, \|y\| = 1} |h(x, y)|$$

and the RHS is $\|S\|_{op}$ and the RHS is $\|h\|_{sesq}$.

Q.E.D.

**Section 3.9. Hilbert-Adjoint Operator.**

**Hilbert-adjoint operator** $T^*$. Let $T : H_1 \to H_2$ be a bounded linear operator, where $H_1$ and $H_2$ are Hilbert spaces. Then the Hilbert-adjoint operator $T^*$ of $T$ is the operator $T^* : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$ $\langle Tx, y \rangle = \langle x, T^*y \rangle$ and $\|T\| = \|T^*\|$.  

**Proof.** Define $h : H_2 \times H_1 \to \mathbb{K}$ by $h(y, x) = \langle y, Tx \rangle$. $h$ has a bounded sesquilinear form. Sesquilinearity is easy, to show boundedness see that 

$$|h(y, x)| = |\langle y, Tx \rangle| \leq \|y\| \cdot \|Tx\| \leq \|y\| \cdot \|x\| \cdot \|T\|_{op}$$

and $T$ is a bounded operator so $\|T\|_{op} < \infty$ verifies the boundedness of this sesquilinear form.

Thus by RR Theorem (adult), $\exists S : H_2 \to H_1$ defined by $h(y, x) = \langle Sy, x \rangle_{H_1}$, so it seems a natural selection to take $T^* = S$.

Next see that 

$$\|T^*\|_{op} = \|S\|_{op} = \|h\|_{sesq} \]
and we want to show that this is \( \|T\|_{op} \). Thus similarly define \( g : H_1 \times H_2 \to \mathbb{K} \) by \( g(x, y) = \langle Tx, y \rangle \implies \exists! \ S : H_1 \to H_2 \) such that \( g(x, y) = \langle Sx, y \rangle \) and therefore \( \langle Sx, y \rangle = \langle Tx, y \rangle \implies \|g\| = \|T\| \).

It is easy to see that \( \|h\| = \|g\| \) by observing

\[
\|g\|_{scesq} = \sup_{\|x\| = 1, \|y\| = 1} |\langle Tx, y \rangle| = \sup_{\|x\| = 1, \|y\| = 1} |\langle y, Tx \rangle| = \|h\|_{scesq}
\]

verifying the norm preservation of the adjoint operator on a Hilbert space.

**Q.E.D.**

**Properties of Hilbert-adjoint operators.** Let \( H_1, H_2 \) be Hilbert spaces, \( S : H_1 \to H_2 \) and \( T : H_1 \to H_2 \) bounded linear operators and \( \alpha \) any scalar. Then we have

\[
\begin{align*}
\langle T^* y, x \rangle &= \langle y, Tx \rangle \\
(S + T)^* &= S^* + T^* \\
(\alpha T)^* &= \bar{\alpha} T^* \\
(T^*)^* &= T \\
\|T^* T\| &= \|TT^*\| = \|T\|^* \\
T^* T = 0 &\iff T = 0 \\
(ST)^* &= T^* S^*
\end{align*}
\]

**Section 3.10. Self-Adjoint, Unitary and Normal Operators.**

**Self-adjoint, unitary and normal operators.** A bounded linear operator \( T : H \to H \) on a Hilbert space \( H \) is said to be

- **self-adjoint or Hermitian** if \( T^* = T \)
- **unitary** if \( T \) is bijective and \( TT^* = T^* T = I \)
- **normal** if \( TT^* = T^* T \)

**Theorem (Self-adjointness).** Let \( T : H \to H \) be a bounded linear operator on a Hilbert space \( H \). Then:

(a) \( T \) is self-adjoint \( \implies \langle Tx, x \rangle \in \mathbb{R} \) for all \( x \in H \)

(b) \( H \) complex and \( \langle Tx, x \rangle \in \mathbb{R} \) for all \( x \in H \) \( \implies T \) self-adjoint

**Theorem (Sequences of self-adjoint operators).** Let \( (T_n) \) be a sequence of bounded self-adjoint linear operators \( T_n : H \to H \) on Hilbert \( H \). Suppose \( (T_n) \) converges, \( T_n \to T \) (i.e. \( \|T_n - T\| \to 0 \) where \( \| \cdot \| \) is the norm on \( B(H, H) \)). Then \( T \) is also self-adjoint.

**Section 4.2. Hahn-Banach Theorem.**

**Hahn-Banach Theorem (baby).** \( X \) vector space over \( \mathbb{K} = \mathbb{R} \), \( Z \) proper subspace of \( X \). \( f : Z \to \mathbb{R} \) is a linear functional such that \( f \leq p \) where \( p \) is sub-linear (i.e. \( p(\alpha x) = \alpha p(x), \alpha \geq 0 \) and \( p(x + y) \leq p(x) + p(y) \)) \( \implies \exists \bar{f} : X \to \mathbb{R} \) linear functional such that \( \bar{f} |_Z = f \) and \( \bar{f} \leq p \).

**Zorn’s Lemma.** \( M \) partially ordered \((\leq)\) set, i.e. (1) \( a \leq a \), (2) \( a \leq b, b \leq a \implies a = b \), and (3) \( a \leq b, b \leq c \implies a \leq c \), and any chain (totally ordered subset) has an upper bound \( \implies \exists \) maximal element in \( M \).
Proof (HB baby). Define

\[ M = \{ g : D(g) \to \mathbb{R} \mid g \text{ is linear functional}, Z \subseteq D(g) \subseteq X, g|_Z = f, g \leq p \} \]

This is a partially ordered set under the ordering of \( g_1 \leq g_2 \iff D(g_1) \subseteq D(g_2) \) and \( g_2|_{D(g_1)} = g_1 \). Any chain \( C \subseteq M \) has an upper bound given by

\[ \hat{g}(x) = g(x) \text{ if } x \in D(g) \text{ for any } g \in C \]

which is clearly a linear functional with domain \( D(\hat{g}) = \bigcup_{g \in C} D(g) \).

Clearly \( \hat{g} \) is an upper bound since by definition and construction we have \( g \leq \hat{g} \) for all \( g \in C \). Then there exists a maximal element \( \hat{f} \) in \( M \) satisfying \( \hat{f} \leq p \) and \( f|_Z = f \). We want to show that \( D(\hat{f}) = X \). We have that \( D(\hat{f}) \subseteq X \) so we must show that \( D(\hat{f}) \supseteq X \). For contradiction assume that latter does not hold.

Then \( \exists y_1 \in X - D(\hat{f}) \) and consider \( Y_1 = \text{span}(D(\hat{f}), y_1) \). Note \( y_1 \neq 0 \) since \( 0 \in Z \subseteq D(\hat{f}) \) and \( y_1 \notin X - D(\hat{f}) \).

Then for any \( x \in Y_1 \) we have \( x = y + \alpha y_1 \) for some \( y \in D(\hat{f}) \). Note that this representation must be unique as if we have \( x = y' + \alpha' y_1 \) then \( y' + \alpha' y_1 = y + \alpha y_1 \iff y - y' = (\alpha' - \alpha)y_1 \) and the LHS is in \( D(\hat{f}) \) and thus since \( y_1 \notin D(\hat{f}) \) then \( \alpha' - \alpha = 0 \implies \alpha' = \alpha \) and thus \( y = y' \) showing this representation is unique.

Thus define \( g_1 \) on \( Y_1 \) by \( g_1(y + \alpha y_1) = \hat{f}(y) + \alpha c \) where \( c \in \mathbb{R} \). Clearly this is linear. For \( \alpha = 0 \) then \( g_1 = \hat{f} \).

Then \( g_1 \) is a proper extension of \( f \), contradicting the maximality of \( \hat{f} \) if \( g_1 \leq p \). See that

\[ g_1(x) = \hat{f}(x) + \alpha c \leq -\alpha p \left( -y_1 - \frac{1}{\alpha} y \right) = p(\alpha y_1 + y) = p(x) \]

providing our contradiction.

Q.E.D.

Hahn-Banach Theorem (adult). \( Z \) is a subspace of vector space \( X \), \( f : Z \to \mathbb{K} \) is \( \mathbb{K} \)-linear functional and \( |f| \leq p \) where \( p \) sub-linear (\( p(x+y) \leq p(x) + p(y) \)) and \( p(\alpha x) = |\alpha| p(x) \) for all \( \alpha \in \mathbb{R} \) \( \implies \exists \hat{f} : X \to \mathbb{K} \) is \( \mathbb{K} \)-linear functional such that \( |\hat{f}| \leq p \) and \( \hat{f} |_Z = f \).

Note. \( p(0) = 0 \) and \( p(x) + p(-x) \geq p(x + (-x)) = p(0) = 0 \implies p(x) \geq 0 \) for any \( x \in X \).

Proof (HB adult). \( f : Z \to \mathbb{C} \) such that \( f(x) = f_1(x) + if_2(x) \) where \( f_1, f_2 : Z \to \mathbb{R} \) are linear functionals. Note that \( f_2 \) is uniquely determined by \( f_1 \) defined by \( f_2(x) = -f_1(ix) \). We can show this by matching real parts in the following equalities

\[ f(ix) = f_1(ix) + if_2(ix) \]
\[ if(x) = if_1(x) - f_2(x) \]

Therefore

\[ f(x) = f_1(x) - if_1(ix) \]

Use HB baby on \( f_1 \) to extend \( f_1 \) (note \( |f_1| \leq |f| \leq p \)) to \( \bar{f}_1 : X \to \mathbb{R} \). Naturally define

\[ \bar{f}(x) = \bar{f}_1(x) - i\bar{f}_1(ix) \]

which is trivially a \( \mathbb{C} \)-linear functional, clearly \( \bar{f} |_Z = f \) and we need to show that \( |\bar{f}| \leq p \). Note that for any \( z \in \mathbb{C} \) we have \( z = re^{i\theta} \) and thus \( f(x) = |f(x)|e^{i\theta} \) and therefore

\[ |\bar{f}(x)| = |\bar{f}(x)|e^{-i\theta} = \bar{f}(e^{-i\theta}x) = \bar{f}_1(e^{-i\theta}x) - i\bar{f}_1(i(e^{-i\theta}x) \]

and since the LHS is a real number then the imaginary part of the RHS must be 0. Then

\[ 0 \leq |\bar{f}(x)| = \bar{f}_1(e^{-i\theta}x) \leq p(e^{-i\theta}x) = |e^{-i\theta}|p(x) = p(x) \]

verifying the boundedness by the sub-linear functional.
HB Application 1. Let \( Z \) be a subspace of \( X \) a normed vector space, \( f : Z \to \mathbb{K} \) is a \( \mathbb{K} \)-linear functional and bounded \( \implies \exists \tilde{f} : X \to \mathbb{K} \), bounded \( \mathbb{K} \)-linear functional such that \( \tilde{f} |_Z = f \) and \( \|\tilde{f}\| = \|f\| \).

**Proof.** Define \( p : X \to \mathbb{R} \) by \( p(x) = \|f\| \cdot \|x\| \) which is clearly sub-linear (note \( \|f\| < \infty \) and it exists since \( f \) is bounded). Use HB Theorem (adult) so then \( \exists \tilde{f} : X \to \mathbb{K} \) a \( \mathbb{K} \)-linear functional such that \( \tilde{f} |_Z = f \) and \( |\tilde{f}| \leq p \).

Bounded? Note \( |\tilde{f}(x)| \leq p(x) = \|f\| \cdot \|x\| \implies \|\tilde{f}\| \leq \|f\| < \infty \).

Equality of norms? Note that we have \( \leq \) above so we WTS \( \geq \). See that

\[
\|\tilde{f}\| = \sup_{x \in X - \{0\}} \frac{|\tilde{f}(x)|}{\|x\|} \geq \sup_{x \in Z - \{0\}} \frac{|\tilde{f}(x)|}{\|x\|} = \sup_{x \in Z - \{0\}} \frac{|f(x)|}{\|x\|} = \|f\|
\]

verifying the equality using the boundedness above.

Q.E.D.

HB Application 2. \( X \) normed vector space, \( x \in X \). \( X' \) is space of bounded linear functionals \( f : X \to \mathbb{K} \) (\( \mathbb{K} \)-linear functional). Fix \( x \in X \), then

\( \bar{x} : X' \to \mathbb{K} \) defined by \( \bar{x}(f) = f(x) \)

Further, \( \|x\|_X = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{\|f\|_{op}} = \|\bar{x}\|_{op} \).

**Proof.** Note \( \|\bar{x}\|_{op} \leq \|x\|_X \) since \( |f(x)| \leq \|f\| \cdot \|x\| \implies \frac{|f(x)|}{\|f\|_{op}} \leq \|x\| \implies \|\bar{x}\|_{op} = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{\|f\|_{op}} \leq \|x\|_X \).

Now we want to show that \( \|\bar{x}\|_{op} \geq \|x\|_X \). Construct \( Z = \text{span}\{x\} = \{\alpha x \mid \alpha \in \mathbb{K}\} \). Let \( g : Z \to \mathbb{K} \) be such that \( g(\alpha x) = \alpha \cdot \|x\| \) and it is easy to see this is a linear functional on \( Z \) that is bounded because \( |g(\alpha x)| = |\alpha| \cdot \|x\| = |\alpha| \|x\| \implies \|g\| = \sup_{x \in Z} \frac{|g(x)|}{\|x\|} = \sup_{z = \alpha x \in Z - \{0\}} \frac{\|\alpha x\|}{\|x\|} = 1 \) proving \( \|g\| = 1 \).

Thus \( \exists \bar{g} : X \to \mathbb{K} \) is bounded linear functional such that \( \bar{g} |_Z = g \) and \( \|\bar{g}\| = \|g\| = 1 \). Then

\[
\|\bar{x}\|_{op} = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{\|f\|_{op}} \geq \frac{|\bar{g}(x)|}{\|\bar{g}\|} = \frac{\|x\|}{1} = \|x\|
\]

Therefore \( \|\bar{x}\|_{op} \geq \|x\|_X \) verifying the equality.

Q.E.D.

**Adjoint operator.** \( T : H_1 \to H_2 \) are Hilbert spaces \( \implies \exists T^* : H_2 \to H_1 \) such that \( \langle Tx, y \rangle = \langle x, T^*y \rangle \) and \( \|T^*\| = \|T\| \).

HB Application 3 (Adjoint operator). \( T : X \to Y \) where \( X, Y \) normed vector spaces and \( T \) is a bounded linear operator \( \implies \exists T^* : Y' \to X' \) bounded linear operator with \( \|T^*\| = \|T\| \).

**Proof.** Define \( T^* : Y' \to X' \) by \( g \in Y' \to T^*g \in X' \) where \( T^*g : X \to \mathbb{K} \) is defined by \( \langle T^*g, x \rangle \). We want to check this is linear, bounded, and preserves the norm of \( T \).

“Bounded.” \( |\langle T^*g, x \rangle| = |\langle g(Tx), x \rangle| \leq \|g\| \|Tx\| \leq \|g\| \|T\| \|x\| \) using the boundedness of \( g \) and \( T \).

“Linear.” Cleary \( T^*g \) as a functional on \( X \) is linear from the linearity of \( T \) and \( g \). \( T^*g \)’s linearity follows.

“Norm equality.” First see that

\[
\|T^*g\|_{op} = \sup_{x \in X - \{0\}} \frac{|\langle T^*g, x \rangle|}{\|x\|_{X}} = \sup_{x \in X - \{0\}} \frac{|g(Tx)|}{\|x\|_{X}} \leq \sup_{x \in X - \{0\}} \frac{|g| \cdot \|T\| \cdot \|x\|}{\|x\|_{X}} = \|g\| \cdot \|T\|
\]
Assume \( \epsilon \bar{n} \in \emptyset \). Now using \( n \).

**Claim:**

\[
\mathcal{T}_1 = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n \]

\[= \mathcal{A} \]

\[
\bar{A}_i \geq X \implies (\bar{A}_i)^C \neq \emptyset \quad \text{and} \quad (\bar{A}_i)^C \text{ is open} \implies \exists \epsilon_1 > 0, x_1 \in (\bar{A}_i)^C \text{ such that } K_1 = B(x_1, \epsilon_1) \subseteq (\bar{A}_i)^C.
\]

\[
\bar{A}_2 \geq X \implies (\bar{A}_2)^C \neq \emptyset \quad \text{and} \quad (\bar{A}_2)^C \text{ is open and further } B(x_1, \frac{\epsilon_1}{2}) \subseteq \bar{A}_2 \implies K_2 = B(x_1, \frac{\epsilon_1}{2}) \cap (\bar{A}_2)^C \neq \emptyset \quad \text{and is open} \implies \exists \epsilon_2 > 0, x_2 \in K_2 \text{ such that } B(x_2, \epsilon_2) \subseteq K_2 \text{ (note } \epsilon_2 \leq \frac{\epsilon_1}{2} \leq \epsilon_1).\]

\[
\vdots
\]

\[
\bar{A}_{n+1} \geq X \implies (\bar{A}_{n+1})^C \neq \emptyset \quad \text{and is open and } K_{n+1} = B(x_n, \frac{\epsilon_n}{2}) \cap (\bar{A}_{n+1})^C \neq \emptyset \quad \text{and open} \implies \exists \epsilon_{n+1} > 0, x_{n+1} \in K_{n+1} \text{ such that } B(x_{n+1}, \epsilon_{n+1}) \subseteq K_{n+1} \text{ (note } \epsilon_n \leq \frac{\epsilon_{n+1}}{2}).
\]

\[
\vdots
\]

**Claim:** \( \{x_n\}_{n \in \mathbb{N}} \) are Cauchy. Note these “balls” are getting smaller and smaller, they form a nested sequence:

\[
\cdots \subseteq B(x_{n+1}, \epsilon_{n+1}) \subseteq B(x_n, \epsilon_n) \subseteq \cdots
\]

Assume \( m \geq n \implies x_m \in B(x_m, \epsilon_m) \subseteq \cdots \subseteq B(x_n, \frac{\epsilon_n}{2}) \implies x_m \in B(x_n, \frac{\epsilon_n}{2}) \implies d(x_n, x_m) < \frac{\epsilon_n}{2} \leq \frac{\epsilon_n}{2n} \to 0 \text{ since } \epsilon_n \leq \frac{\epsilon_{n+1}}{2n} \text{ as } n \uparrow \infty. \text{ Thus } \{x_n\}_{n \in \mathbb{N}} \text{ is Cauchy and thus converges, } x_n \to x \in X.
\]

Now using \( d(x_n, x_m) < \frac{\epsilon_n}{2} \implies \text{ taking } m \uparrow \infty \text{ we have } d(x_n, x_m) \to d(x_n, x) \leq \frac{\epsilon_n}{2} \implies x \in B(x_n, \epsilon_n) \subseteq (\bar{A}_n)^C \text{ by construction} \implies x \in (\bar{A}_n)^C \text{ for all } n \in \mathbb{N} \implies x \in \bigcap_{n \in \mathbb{N}} (\bar{A}_n)^C = \left( \bigcup_{n \in \mathbb{N}} \bar{A}_n \right)^C = X^C = \emptyset \implies x \in \emptyset \text{ gives our contradiction.}
\]

Q.E.D.
**BC Theorem Application.** \(T_n\) bounded linear operator, \(T_n : X \to Y\) where \(X\) Banach and \(Y\) normed vector space \(\implies\) \(\sup_{n \in \mathbb{N}} \|T_n\|_{op} < \infty\).

**Proof.** \(T_n\) bounded \(\implies\) \(\|T_n\| < \infty\). Let \(x \in X\) \(\implies\) \(\exists k \in \mathbb{N} \ni \sup_{n \in \mathbb{N}} \|T_n x\| \leq k\). Then for arbitrary \(k \in \mathbb{N}\) we have

\[
A_k = \left\{ x \in X \mid \sup_{n \in \mathbb{N}} \|T_n x\| \leq k \right\} \implies x \in \bigcup_{k \in \mathbb{N}} A_k
\]

and thus \(X = \bigcup_{k \in \mathbb{N}} A_k\) and since \(X\) is of second category, \(\exists k_0 \in \mathbb{N}\) such that \(A_{k_0}\) is nowhere dense. That is, \(A_{k_0} \supseteq\) an open set \(\implies\) \(B(x_0, \epsilon_0) \subseteq A_{k_0} \implies \|x - x_0\| < \epsilon_0 \implies x \in A_{k_0} \implies \sup_{n \in \mathbb{N}} \|T_n x\| \leq k_0\).

Using \(\|T_n\|_{op} = \sup_{u \in S_X(0, 1)} \|T_n u\|_X\). Note that for \(u \in S_X(0, 1)\) and \(\epsilon < \epsilon_0\) we have \(x_0 + \epsilon u \in B(x_0, \epsilon_0) \subseteq A_{k_0} \implies \sup_{n \in \mathbb{N}} \|T_n(x_0 + \epsilon u)\|_Y \leq k_0\) and thus

\[
\epsilon \|T_n u\| - \|T_n x_0\| \leq \|T_n x_0 + \epsilon T_n u\|_Y \leq k_0 \implies \epsilon \|T_n u\| \leq k_0 + \|T_n x_0\| \leq k_0 + k_0 = 2k_0
\]

and therefore \(\|T_n u\| \leq \frac{2k_0}{\epsilon}\) and since \(k_0, \epsilon\) were fixed then

\[
\sup_{u \in S_X(0, 1)} \|T_n u\| \leq \frac{2k_0}{\epsilon} \implies \|T_n\|_{op} \leq \frac{2k_0}{\epsilon} \implies \sup_{n \in \mathbb{N}} \|T_n\|_{op} \leq \frac{2k_0}{\epsilon}
\]

Q.E.D.