General

**Logarithmic** - \( f(z) = \log z = \ln |z| + \arg z \) and \( f(z) = \Log z = \ln |z| + \Arg z \)

**Trigonometric Functions** - \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \) and \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \) and \( \tan z = \frac{\sin z}{\cos z} \)

**Hyperbolic Functions** - \( \sinh z = \frac{e^{z} - e^{-z}}{2} \) and \( \cosh z = \frac{e^{z} + e^{-z}}{2} \)

**Length of Path** - \( L = \int_{a}^{b} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt = |z'(t)| \, dt \)

**Line Integral** - \( \int_{C} f(z) \, dz = \int_{a}^{b} f(z(t)) z'(t) \, dt \)

**Theorem on Bound of Integral** - If \( C \) is a contour of length \( L \) and \( f \) is a piecewise continuous function on \( \mathbb{C} \). If we assume \(|f(z)| \leq M \forall z \in \mathbb{C} \), then

\[
\left| \int_{C} f(z) \right| \leq M \cdot L
\]

**Cauchy-Goursat Theorem** - If a function \( f \) is analytic at all points interior to and on a simple closed contour \( C \), then

\[
\int_{C} f(z) \, dz = 0
\]

**Cauchy Integral Theorem** - \( f \) is analytic everywhere inside and on simple closed contour \( C \), in positive sense. If \( z_0 \) is interior to \( C \), then

\[
f(z_0) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} \, dz
\]

and this can be extended to

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C} \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \quad n = 1, 2, \ldots
\]

**Theorem** - If a function \( f \) is entire and bounded in the complex plane, then \( f(z) \) is constant throughout the plane.

**Series**

**Taylor Series Theorem** - An analytic function \( f \) throughout a disk \(|z - z_0| < R_0 \) centered at \( z_0 \) and with radius \( R_0 \) has a unique power series representation

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad , \quad a_n = \frac{f^{(n)}(z_0)}{n!}
\]
Common Series

Good formulas to know:

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

\[ \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \]

\[ \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \]

\[ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1) \]

**Laurent Series Theorem** - Suppose that a function \( f \) is analytic throughout an annular domain \( R_1 < |z - z_0| < R_2 \), centered at \( z_0 \), and let \( C \) denote any positively oriented simple closed contour around \( z_0 \) and lying in that domain. Then, at each point in the domain, \( f(z) \) has the series representation

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \]

where

\[ a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots) \]

\[ b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots) \]

**Residues and Poles**

**Cauchy’s Residue Theorem** - Let \( C \) be a simple closed contour, described in the positive sense. If a function \( f \) is analytic inside and on \( C \) except for a finite number of singular points \( z_k \) (\( k = 1, 2, \ldots, n \)) inside \( C \), then

\[ \int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) \]

**Residue at Infinity** - Residue at infinity is given by

\[ \text{Res}_{z=\infty} f(z) = \text{Res}_{z=0} \left( \frac{1}{z} \right) f\left( \frac{1}{z} \right) \]

and we can use this in the formula

\[ \int_C f(z)dz = 2\pi i \sum_{z=\infty} \text{Res}_{z=0} \left( \frac{1}{z} \right) f\left( \frac{1}{z} \right) \]

**Residue Theorem 1** - An isolated singular point \( z_0 \) of a function \( f \) is a pole of order \( m \) if and only if \( f(z) \) can be written in the form

\[ f(z) = \frac{\phi(z)}{(z - z_0)^m} \]

where \( \phi(z) \) is analytic and nonzero and \( z_0 \). Moreover,

\[ \text{Res}_{z=0} f(z) = \phi(z_0) \quad \text{if } m = 1 \]
and
\[ \text{Res}_{z=z_0} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if} \quad m \geq 2 \]

**Residue Theorem 2** - Let two functions \( p \) and \( q \) be analytic at a point \( z_0 \). If
\[ p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0 \]
then \( z_0 \) is a simple pole of the quotient \( p(z)/q(z) \) and
\[ \text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \]

**Applications of Residues**

**Cauchy Principal Value** - is given by
\[ \text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx \]

**Evaluation of Improper Integrals**

Steps to evaluate an integral \( \int_{0}^{\infty} f(x)dx \) where \( f \) is even:

1. Draw a contour from \((-R,0)\) to \((R,0)\) (to the right) and then a semi-circle from \((R,0)\) to \((-R,0)\) counter-clockwise.

2. This is a closed contour and we can write
\[ \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=0}^{n} \text{Res}_{z=z_k} f(z) \]
where each \( z_k \) \((k = 0, 1, \ldots, n)\) are isolated singularities in the upper half-plane.

3. Look at when \(|z| = R\) and show that \(|f(z)|\) is bounded by \( M_R \). Use this to show that
\[ \left| \int_{C_R} f(z)dz \right| \leq M_R \cdot \pi R \to 0 \implies \int_{C_R} f(z)dz \to 0 \quad \text{as} \quad R \to \infty \]

4. Let \( R \to \infty \) in 2. above and thus we have shown that
\[ \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) \]
and using that \( f \) is even we see
\[ \int_{0}^{\infty} f(x)dx = \pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) \]

**Evaluation of Improper Integrals Using Indented Paths**

**Jordan’s Lemma** - Suppose that

- a function \( f(z) \) is analytic at all points in the upper half plane \( y \geq 0 \) that are exterior to a circle
\[ |z| = R_0 \]
• $C_R$ denotes a semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$, where $R > R_0$

• for all points $z$ on $C_R$ there is a positive constant $M_R$ such that

$$|f(z)| \leq M_R \to 0 \quad \text{as } R \to \infty$$

Then for every positive constant $a$,

$$\lim_{R \to \infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

**Indented Paths** - Use when $f(z)$ is not analytic at $z = 0$ and use the fact that

$$\lim_{\rho \to 0} \int_{C_{\rho}} f(z)dz = -\pi i \cdot \text{Res}_{z=0} f(z)$$

and also use when $f(z)$ involving $\log z$ yet just show

$$\lim_{\rho \to 0} \int_{C_{\rho}} f(z)dz = 0$$

**Definite Integrals Involving Sines and Cosines**

Evaluating integrals such as

$$\int_{0}^{2\pi} F(\sin \theta, \cos \theta)d\theta$$

can be done by looking at the circle $|z| = 1$ and converting this integral to an integral about that contour using the substitutions

$$\sin \theta = \frac{z - z^{-1}}{2i}, \quad \cos \theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{iz}$$

**Rouche’s Theorem**

Let $C$ denote a simple closed contour and suppose that

• two functions $f(z)$ and $g(z)$ are analytic inside $C$

• $|f(z)| \geq |g(z)|$ at each point on $C$

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside $C$.  

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