MA 513
Study Guide

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1 Basic Definitions

Given $z = x + iy$ we define Re$z = x$ and Im$z = y$. We also define $\bar{z} = \overline{x + iy} = x - iy$.

2 Basic Theorems

$|z_1 + z_2| \leq |z_1| + |z_2|$ and $||z_1| - |z_2|| \leq |z_1 - z_2|$

3 Limits

Definition - Let a function f be defined at all points z in some deleted neighborhood of z_0. The statement that the limit of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0$$

means that $\forall\ \epsilon > 0\ \exists\ \delta\ \ni$

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

3.1 Limits Involving the Point at Infinity

$$\lim_{z \to z_0} f(z) = \infty \qquad \text{if and only if} \qquad \lim_{z \to z_0} \frac{1}{f(z)} = 0 \tag{1}$$

$$\lim_{z \to \infty} f(z) = w_0 \qquad \text{if and only if} \qquad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0 \tag{2}$$

$$\lim_{z \to \infty} f(z) = \infty \qquad \text{if and only if} \qquad \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0 \tag{3}$$

Theorem on Limits (like L'Hobital's) - Suppose $f(z), g(z)$ are differentiable at z_0, $f(z_0) = g(z_0) = 0$, $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

4 Topology

Interior point - z_0 is an interior point of a set S if $\exists\, \epsilon > 0 \ni B(z_0, \epsilon) = \{z \,||z - z_0| < \epsilon\} \subset S$.

Exterior point - z_0 is an exterior point of a set S if $\exists\, \epsilon > 0 \ni B(z_0, \epsilon) \cap S = \emptyset$

Boundary point - z_0 is neither of the above

Open set - a set that contains no boundary points

Closed set - a set that contains all of its boundary points OR a set that contains all its accumulation points

Accumulation point - z_0 is an accumulation point if every open neighborhood about z_0 contains points in S other than z_0

5 Differentiability

Definition

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0. The **derivative** of f at z_0 is the limit:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists.

We can also write:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Rectangular Coordinates

Given $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$, f is differentiable when the first order partials exist in a neighborhood of z and they are continuous at the point and the Cauchy-Riemann equations are satisfied:

$$u_x = v_y$$
$$u_y = -v_x$$

then $f'(z) = u_x + iv_x$.

Polar Coordinates

Given $f(z) = u(r,\theta) + iv(r,\theta)$ where $z = re^{i\theta}$, f is differentiable when the first order partials exist in a neighborhood of z and they are continuous at the point and the Cauchy-Riemann equations are satisfied:

$$ru_r = v_\theta$$
$$u_\theta = -ru_r$$

then $f'(z) = e^{-i\theta}(u_r + iv_r)$.

Analytic Functions

Lemma - If f is analytic on D and $f(z) = 0$ at each point of a line segment on D, then $f(z) \equiv 0$ on D

Reflection Principle - If f is analytic on D containing a segment of the real axis and D is symmetric about the axis, then

$$\overline{f(z)} = f(\bar{z})$$

$\iff$ at each point $x \in$ segment, $f(z)$ is real.

6 Elementary Functions

Exponential - $f(z) = e^z = e^x(\cos y + i \sin y)$

Logarithmic - $f(z) = \log z = \ln|z| + \arg z$ and $f(z) = \text{Log} z = \ln|z| + \text{Arg} z$

Trigonometric Functions - $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\tan z = \frac{\sin z}{\cos z}$

Hyperbolic Functions - $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z+e^{-z}}{2}$

Inverse Trig. Functions - $\sin^{-1} z = -i \log\left[iz + \left(1-z^2\right)^{\frac{1}{2}}\right]$, $\cos^{-1} z = -i\log\left[z + i\left(1-z^2\right)^{\frac{1}{2}}\right]$, $\tan^{-1} z = \frac{i}{2} \cdot \log\frac{1+z}{1-z}$

We can prove this by letting $w = \sin z$ and looking at $z = \frac{e^{iw}-e^{-iw}}{2i}$ and solving for e^{iw}.

Inverse Hyperbolic Trig Functions - $\sinh^{-1} z = \log\left[z + \left(z^2+1\right)^{\frac{1}{2}}\right]$, $\cosh^{-1} z = \log\left[z + \left(z^2-1\right)^{\frac{1}{2}}\right]$, $\tanh^{-1} = \frac{1}{2}\cdot\log\frac{1+z}{1-z}$

7 Integration

Length of Path - $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_a^b |z'(t)|\,dt$

Line Integral - $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$

Theorem on Bound of Integral - C is a contour of length L and f is a piecewise continuous function on $\mathbb{C}$. If we assume $|f(z)| \leq M \,\forall\, z \in \mathbb{C}$, then

$$\left|\int_C f(z)\right| \leq M \cdot L$$

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Cauchy-Goursat Theorem

Theorem - Let $f(z)$ be analytic at all points interior to and on a closed contour C, then

$$\int_C f(z)dz = 0$$

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Corollary - $C, C_1, C_2, \ldots, C_k$ are simply connected closed contours with each C_i interior to C such that C is oriented counter-clockwise and each C_i is clockwise. If $f(z)$ is analytic on each C_i and C and also at all points in the multiply connected domain, then

$$\int_C f(z)dz + \sum_{i=1}^{k}\int_{C_i} f(z)dz = 0$$

Corollary - $C_2 \subset C_1$ and $f(z)$ is analytic on and between the two, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Cauchy Integral Formula

Theorem - f is analytic everywhere inside and on simple closed contour C, in positive sense. If z_0 is interior to C, then

$$f\left(z_0\right)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-z_0}dz$$

and this can be extended to

$$f^{(n)}\left(z_0\right)=\frac{n!}{2\pi i}\int_C\frac{f(z)}{\left(z-z_0\right)^{n+1}}dz,\qquad n=1,2,\ldots$$