Notes - Sylow Theorems

MA 407H - Szanto

Definition: *Conjugacy Class of a*
Let a be an element in a group G. The *conjugacy class of* a is the set denoted by $\mathrm{cl}(a)$ and defined by $\mathrm{cl}(a) = \{ \ xax^{-1} \mid x \in G \ \}$.

Theorems and Corollaries
Let G be a finite group and let $a \in G$. Remember $C(a)$ denotes the centralizer of a in G. Let p be prime and $k \in \mathbb{N}$.

- $|\mathrm{cl}(a)| = |G : C(a)|$
- $|\mathrm{cl}(a)|$ divides $|G|$
- $|G| = \sum |G : C(a)| = \sum |\mathrm{cl}(a)|$ for one element a from each conjugacy class
- $|G| = p^k$ for some prime p and $k \in \mathbb{N} \implies |Z(G)| > 1$.
- $|G| = p^2 \implies G$ is Abelian.

Definition: *Sylow p-Subgroup*
Let G be a finite group and let p be a prime divisor of $|G|$. If p^k divides $|G|$ but p^{k+1} does not, then any subgroup of G of order p^k is called a *Sylow p-subgroup of G*.

Definition: *Conjugate Subgroups*
Let H and K be subgroups of a group G. We say that H and K and *conjugate* if there is a $g \in G$ such that $H = gKg^{-1}$.

Sylow Theorems
Let G be a finite group and let p be prime. Let $k \in \mathbb{N}$.

1. p^k divides $|G| \implies G$ has at least one subgroup of order p^k.
2. If H is a subgroup of G and $|H|$ is a power of p, then H is contained in some Sylow p-subgroup of G.
3. The number of Sylow p-subgroups is equal to $1 \mod p$ and divides $|G|$. Also, any two Sylow p-subgroups of G are conjugate.

Theorems and Corollaries
Suppose G is a finite group.

- Let p be a prime that divides $|G|$. Then G has an element of order p.
- A Sylow p-subgroup of G is normal $\iff$ it is the only Sylow p-subgroup.
- Let $|G| = pq$ for primes p, q with $p < q$. p does not divide $q - 1 \implies G$ is cyclic. Furthermore, $G \cong Z_{pq}$.