Math 1110: Solutions for 2.6 Practice Problems

Problem 1  Consider the function $f(x) = \frac{x^2}{x^2 - 4}$

(a) Is $f(x)$ an even function, an odd function of neither? Please justify your answer.

Answer: $f(x)$ is an even function since $f(-x) = \frac{(-x)^2}{(-x)^2 - 4} = \frac{x^2}{x^2 - 4} = f(x)$

(b) Compute $\lim_{x \to \infty} f(x)$. What does this limit tell you about the asymptotes of the graph of $f(x)$?

$\lim_{x \to \infty} \frac{x^2}{x^2 - 4} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2}}{\frac{x^2 - 4}{x^2}} = \lim_{x \to \infty} \frac{1}{1 - \frac{4}{x^2}} = \frac{1}{1 - 0} = 1$

Problem 2  Evaluate the following expressions, or explain why you cannot. You may only use techniques that we have learned in Chapters 1 and 2.

(a) $\lim_{x \to \infty} (\sqrt{x^2 + 12x} - x) = \lim_{x \to \infty} (\sqrt{x^2 + 12x} - x) \cdot \frac{\sqrt{x^2 + 12x} + x}{\sqrt{x^2 + 12x} + x}$

$= \lim_{x \to \infty} \frac{x^2 + 12x - x^2}{\sqrt{x^2 + 12x} + x}$

$= \lim_{x \to \infty} \frac{12x}{\sqrt{x^2 + 12x} + x}$

$= \lim_{x \to \infty} \frac{12}{\sqrt{x^2 + 12x} + x}$

$= \lim_{x \to \infty} \frac{12}{\sqrt{1 + \frac{12}{x}} + 1}$

$= \frac{12}{\sqrt{1} + 1} = 6$

(b) $\lim_{\theta \to 0} \sqrt{1 + \csc \theta} = \lim_{\theta \to 0} \sqrt{1 + \frac{1}{\sin \theta}}$

This limit does not exist because $\lim_{\theta \to 0^+} \sqrt{1 + \frac{1}{\sin \theta}} = \infty$ and $\lim_{\theta \to 0^-} \sqrt{1 + \frac{1}{\sin \theta}}$ is undefined. When $\theta$ approaches zero from the left, the inside of the square root is negative, so the function is undefined, which makes the left hand limit undefined, and the overall limit undefined as well.

(c) $\lim_{x \to 0^+} \frac{x^2 - 3x + 2}{x^3 - 2x^2} = \lim_{x \to 0^+} \frac{(x - 1)(x - 2)}{x^2(x - 2)}$

The numerator of this function approaches 2 (a positive number) as $x$ approaches 0, and the denominator approaches zero, but will be negative overall, so the whole function is negative and does not exist because it goes to $-\infty$. 
(d) \[
\lim_{x \to \infty} \frac{x^5 + 7x^2 - 4 + x^5}{x^5 - x + 1} = \lim_{x \to \infty} \frac{1 + 7/x^3 - 4/x^5 + 1}{1 - 1/x^4 + 1/x^5} = 1 + 0 - 0 + 1 = 2
\]

Problem 3 2.6: 29- See solutions posted on Moodle

Problem 4 2.6: 57- See solutions posted on Moodle

Problem 5 2.6: 73- See solutions posted on Moodle

Problem 6 True or False. More questions on 2.5

- **True or False**: At some time since you were born your weight in pounds equaled your height in inches.
  
  **True**: Let \( f(t) \) be your height minus your weight at any point of time \( t \) in years. When you were you probably weighed about 8 pounds. You were probably about 21 inches long, so \( f(0) \approx 21 - 8 = 13 \). Now you weight a lot more than your height in inches (for example, an 21 year old adult could reasonably weigh 145 pounds and be 67 inches tall, so \( f(21) = 67 - 145 = -78 \) so at some point in time \( f(t) = 0 \) (by IVT) and at this time your height equaled your weight.

- **True or False**: Along the Equator, there are two diametrically opposite sites that have exactly the same temperature at the same time.
  
  **True**: Consider any point \( A \) on the equator. Then consider point \( B \) is the place on the Equator that is diametrically opposite of point \( A \). Let \( f(A,B) \) be the temperature at \( A \) minus the temperature at \( B \). Then think about moving the first point around the equator until it is at point \( B \). If \( f(A,B) \) was positive, now \( f(B,A) \) will be negative, and if \( f(A,B) \) was negative, now \( f(B,A) \) will be positive. Either way, there is a change in sign of the function \( f \), so since the context makes it reasonable to assume that \( f \) is continuous, by the intermediate value theorem there must be a point where \( f(X_1, X_2) = 0 \) as we move the first point from \( A \) to \( B \). This is the pair of points that have the same temperature. The only other option is if \( f(A,B) = 0 \) but then the original points had the same temperature and we don’t need to look elsewhere along the equator.

- Suppose that during half-time at a basketball game the score of the home team was 36 points.
  
  **True or False**: There had to be at least one moment in the first half when the home team had exactly 25 points.
  
  **False**: Since the scoring in a basketball game is not a continuous function the intermediate value theorem does not apply. It is entirely possible that the team had 24 points, then scored a 3 pointer and their score jumped to 27 points without ever being 25 points.

- **True or False.** \( x^{100} - 9x^2 + 1 \) has a root in \([0, 2]\).
  
  **True**: Let \( f(x) = x^{100} - 9x^2 + 1 \).
  
  \[
f(0) = 1
\]
  
  \[
f(1) = 1 - 9 + 1 = -9
\]
Since $f(x)$ is a polynomial it is continuous, the intermediate value theorem tells us that there must be a point $c$ in the interval $[0, 1]$ where $f(c) = 0$. $c$ is thus a root of $x^{100} - 9x^2 + 1$ in the desired interval.