1. At time $t$ seconds, the two positions of two particles are coordinates $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocity?

2. 

3. The figure below shows two graphs, a graph of $f(x)$ with its derivative $f'(x)$. Which graph is which and how do you know?

4. Find the derivative of $y = \log_5 (3x - 7)$.
Genna's Test Question Solutions

1. The velocities will be equal when \( S_1'(x) = S_2'(x) \) so we take the derivatives independently and then set them equal to each other:

\[
S_1' = 9t^2 - 24t + 18 \\
S_2' = -3t^2 + 18t - 12
\]

\[
S_1' = S_2' \quad \rightarrow \quad 9t^2 - 24t + 18 = -3t^2 + 18t - 12 \\
+3t^2 \cdot -18t + 18. \\
+3t^2 \cdot -18t + 18.
\]

\[
12t^2 - 42t = -30
\]

\[
2t^2 - 7t = -5
\]

\[
2t^2 - 7t + 5 = 0
\]

\[
x = \frac{7 \pm \sqrt{49 - 40}}{4} = \frac{7 \pm \sqrt{9}}{4} = \frac{7 \pm 3}{4}, \quad t = \frac{1}{4}, \frac{4}{4} \text{ or } \frac{5}{4}.
\]

Therefore, the velocities of the two particles will be equal when \( t = \frac{5}{4} \) and 1 second.
3. We know that the parabolic graph is the derivative of the other graph for a few reasons. The first is that when the pink graph crosses the x-axis, the parabolic graph has a horizontal tangent. We also know this is true because the pink graph is always decreasing, and the derivative of the pink graph is always negative. The derivatives of the parabolic graph start out positive but are constantly decreasing towards zero until they pass zero, become negative, and continue decreasing. The final reason we know the green graph is the derivative of the pink is that logically, we can guess that the pink graph is some shift of the graph $y = -x^3$ and that the green graph is a shift of the graph $y = x^2$.

4. $y = \log_5 (3x-7)$

$$y' = \frac{1}{(3x-7) \ln 5} \cdot \frac{d}{dx} (3x-7)$$

by the chain rule + log rules of differentiation

$$y' = \frac{1}{3x \ln 5 - 7 \ln 5} \cdot 3$$

$$y' = \frac{1}{x \ln 5 - \frac{7}{3} \ln 5}$$
Differentiate the function. Then find the equation of the tangent line at the indicated point.

\[ y = \frac{3x^2}{x} \quad \text{at} \quad (1,3) \]

**Solution 1)**

\[
y = \frac{3x^2}{x}
\]

\[
\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^2}{x}\right)
\]

\[
= 3x^2 \left(\frac{1}{x}\right) - 3x^2 \left(\frac{1}{x^2}\right)
\]

\[
= 3x - \frac{3}{x}
\]

Let \( x = 1 \)

\[
m = \frac{dy}{dx} = 3(1) - \frac{3}{1} = 3(1 - \frac{1}{1}) = 3\left(\frac{1}{1}\right) = 3
\]

\[
y = mx + b \implies y = 3x + b \] to find \( b \)

\[
y = 3(1) + b \implies 3 = a + b \] to find \( a \)

Equation of tangent: \( y = \frac{9}{2}x - \frac{3}{2} \)

Find the derivative of the function.

**Solution 2)**

\[
y = x^2 \cos x + e^{2x} \sin x
\]

\[
\frac{dy}{dx} = \frac{d}{dx}(x^2 \cos x + e^{2x} \sin x)
\]

**Product Rule**

\[
\frac{dy}{dx} = [2x \cos x + x^2 \sin x] + [e^{2x} \cos x + 2e^{2x} \sin x]
\]

\[
= 2x \cos x + x^2 \sin x + e^{2x} \cos x + 2e^{2x} \sin x
\]

A 17ft ladder is leaning against a house when the base starts to slide away. By the time the base equals 8ft from the house, the base is moving at the rate of \( \frac{d}{dt} \) ft/sec.

How fast is the top of the ladder sliding down the wall then?

**Solution 3)**

\[
L = 17
\]

\[
\frac{dL}{dt} = \frac{8}{\sec} \quad 8 \text{ft} \quad \frac{dL}{dt} = \text{change of ladder}
\]

\[
a = \text{house/wall}
\]

\[
b = \text{base}
\]

\[
c = \text{ladder}
\]

Find \( \frac{da}{dt} \) by finding derivative of \( a^2 + b^2 = c^2 \)

\[
\frac{d}{dt}(a^2 + b^2) = c^2
\]

plug in

\[
2(15) \frac{da}{dt} + 2(8)(2) = 2(17)
\]

\[
30 \frac{da}{dt} + 32 = 34
\]

\[
\frac{da}{dt} = \frac{2}{30} = \frac{1}{15} \text{ ft/sec}
\]

The top of the ladder is sliding down at the speed of \( \frac{1}{15} \text{ ft/sec} \).
1. Find the derivative.

\[ Y = x^2 \cot 5x \]
\[ y' = 2x \left( \cot 5x + \frac{x^2}{\csc^2 5x} \right) \cdot 5 \]
\[ y' = 2x \cot 5x - 5x^2 \csc^2 5x \]

2. Are there any points on the curve \( y = \frac{x}{2} + \frac{1}{2x-4} \) where the slope is \(-3/2\)? If so, find them.

\[ \frac{dy}{dx} = \frac{1}{2} - \frac{2}{(2x-4)^2} \cdot 2 \]
\[ \frac{dy}{dx} = \frac{1}{2} - \frac{2}{(2x-4)^2} \]
\[ \frac{-3}{2} = \frac{1}{2} - \frac{2}{(2x-4)^2} \]
\[ -2 = -2 \cdot \frac{2}{(2x-4)^2} \]

3. The radius of a circle is changing at the rate of \(-2/\pi\) m/sec. At what rate is the circle's area changing when \( r = 10 \) m?

\[ \frac{dA}{dt} = \frac{-2}{\pi} \text{ m/sec} \]
\[ A = \pi r^2 \]
\[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \]
\[ \frac{dA}{dt} = 2\pi (10)(\frac{-2}{\pi}) \]
\[ \frac{dA}{dt} = -40 \text{ m}^2/\text{sec} \]
Prelim 2 Practice Questions

1. Find the first derivative with respect to $x$: $f(x) = e^{-\frac{x}{2}} + \ln(x^2 + e^{2x})$
   
   a. $f'(x) = -\frac{1}{2}e^{-\frac{x}{2}} + \frac{2x + 2e^{2x}}{x^2 + e^{2x}}$

2. Derive this equation with respect to $x$: $y = \frac{\sec 2x}{\cos e^x}$
   
   a. $y' = \left\{ \frac{\cos e^x[2(\sec 2x \tan 2x)] + \sec 2xe^x \sin e^x}{(\cos e^x)^2} \right\}$

3. Calculate $y'$ by deriving with respect to $x$: $y^4 + 2x^3 = e^{4x} + \ln(2x + 2) + 2y^4$
   
   a. $y' = \frac{4e^{4x} + \frac{3}{x+1} - 6x^2}{4y^3 - 8y^3}$

4. Find the slope of the circle at the point $(4,6)$: $x^2 + y^2 = 49$
   
   a. $-3/2$

5. Find the derivative of the function with respect to $x$: $g(x) = \sec^{-1} 4x^3$
   
   a. $g'(x) = \frac{3}{x\sqrt{16x^6 - 1}}$
1) Find the constants $a$ & $b$ such that the function

$$f(x) = \begin{cases} -e^{\sin x} & x < 0 \\ ax + b & x \geq 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$.

2) Find the derivative of $|x|$, $x \neq 0$.

3) A line with slope $m$ passes through the origin and is tangent to the graph $y = \ln x$. What is the value of $m$?

4) Find $f''(x)$ if $f(x) = \tan^{-1}(\sqrt{x})$.

5) Prove $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$. 
1. For $x > 0$, $f'(x) = a$
For $x < 0$, $f'(x) = -\cos x e^{\sin x}$, which is defined for all $x$
So $ax + b$ & $-e^{\sin x}$ are differentiable for all $x$

For $x = 0$

the function at $x = 0$ will be differentiable if the derivative from the left = derivative from the right

\[
\begin{align*}
\text{left} & \quad \text{right} \\
(-\cos x)e^{\sin x} & = a \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad a \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad a \\
\quad & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad -1 = a
\end{align*}
\]

the function must be continuous

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(x)
\]

\[
\lim_{x \to 0^+} -e^{\sin x} = \lim_{x \to 0^-} ax + b = a(0) + b
\]

\[-1 = b = b
\]

\[b = -1\]


FOR $f(x)$ to be differentiable

$a = -1$ & $b = -1$

2. \[
\frac{d}{dx} \left| x \right| = \frac{d}{dx} \sqrt{x^2}
\]

\[
\quad = \frac{d}{dx} \left( x^2 \right)^{1/2}
\]

\[
\quad = \frac{1}{2} \cdot 2x \quad \frac{d}{dx} x^2
\]

\[
\quad = \frac{1}{2} \cdot 2x^{1/2} \cdot 2x
\]

\[
\quad = \frac{2x}{2x} \quad \frac{1}{x} \cdot x^{1/2}
\]

\[
\quad = \frac{x}{1} \quad x \neq 0
\]
We know that the points $(0, 0)$ and $(x, \ln x)$ lie on the graph if we assume the tangency point is at $(x, \ln x)$ where $x > 0$.

We can find the slope of the tangent line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \ln x - 0$$

$$m = \frac{\ln x}{x}$$

We know the slope is derivative of $\ln x$:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Therefore we can set these two equal to each other to find the value of $m$:

$$\ln x = \frac{1}{x}$$

$x \ln x = x$

$\ln x = 1$ e both sides

$e^{\ln x} = e$

$x = e$ because $\frac{d}{dx} \ln x = \frac{1}{x}$

$m = \frac{1}{e}$

If $\tan^{-1} \sqrt{x} = \theta$

we know that

$$\tan y = \frac{\sqrt{x}}{1} = \frac{\text{opp}}{\text{adj}}$$

$$\sec^2 y \left(\frac{dy}{dx}\right) = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x} (1 + \tan^2 y)}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x} (1 + \frac{x}{\sqrt{x}})}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x} (1 + x)}$$
Let \( \alpha = \frac{x}{n} \)

therefore \( n = \frac{x}{\alpha} \)

as \( n \to \infty \)
\( \alpha \to 0 \)

plug in \( \alpha \) & \( \frac{x}{\alpha} \)

\[
\lim_{\alpha \to 0} \left(1 + \alpha\right)^{\frac{x}{\alpha}}
\]

by using the number \( e \) as a limit which states

\[
\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e
\]

we can deduce

\[
\lim_{\alpha \to 0} \left[ \left(1 + \alpha\right)^{\frac{1}{\alpha}} \right]^x = \left[ e \right]^x = e^x
\]

therefore

\[
\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x
\]