Problem 1

\[ x^2 + xy + y^2 = 1 \]

a) find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \)

\[
\frac{d}{dx} \left( x^2 + xy + y^2 \right) = \frac{d}{dx} \left( x \right)
\]

\[
2x + \left( x \frac{dy}{dx} + y \frac{d}{dx}x \right) + 2y \frac{dy}{dx} = 0
\]

\[
2x + \left( x \frac{dy}{dx} + y \right) + 2y \frac{dy}{dx} = 0
\]

\[
-2x + y = 2y \frac{dy}{dx} + x \frac{dy}{dx}
\]

\[
-(2x + y) = (2y + x) \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}
\]

b) the tangent line is parallel to \( y = -x \) when slope \( = -1 \)

\[
\text{Slope} = \frac{dy}{dx} = -1 = -\frac{2x + y}{x + 2y}
\]

\[
2x + y = x + 2y
\]

\[
x = y
\]

points on the curve \( \text{when} \ y = x \)

\[
x^2 + x(x) + (x)^2 = 1
\]

\[
3x^2 = 1
\]

\[
x = \pm \frac{\sqrt{3}}{3}, \ y = x
\]

\[
\left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right), \ \left( -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right)
\]

c) the normal line is horizontal when the slope of the tangent line is undefined.

- derivative is undefined when \( x = 2y \)

\[
y = -\frac{x}{2}
\]

on the curve \( x^2 + x \left( \frac{x}{2} \right) + \left( -\frac{x}{2} \right)^2 = 1 \)

\[
x^2 = \frac{x}{2} + \frac{x^2}{4} = 1 \Rightarrow \frac{3}{4} x^2 = 1 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm \frac{\sqrt{3}}{3} = \pm \frac{2}{\sqrt{3}}
\]

\[
\left( \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \ \left( \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\]
Problem 2

\[ xy^2 + 4y - 10 = 2x \]

a) slope at \((1, 2)\)

\[
\frac{d}{dx} (xy^2 + 4y - 10) = \frac{d}{dx} (2x)
\]

\[
(x \frac{dy}{dx} y^2 + y^2 \frac{dx}{dx}) + \frac{dy}{dx} (4y) - \frac{d}{dx} (-10) = \frac{d}{dx} (2x)
\]

\[
x \cdot 2y \frac{dy}{dx} + y^2 + 4 \frac{dy}{dx} = 2
\]

\[
(2xy + 4) \frac{dy}{dx} = 2 - y^2
\]

\[
\frac{dy}{dx} = \frac{2 - y^2}{2xy + 4}
\]

at \((1, 2)\):

\[
\frac{dy}{dx} = \frac{2 - 4}{2 \cdot 1 + 4} = \frac{-2}{8} = -\frac{1}{4}
\]

b) normal line = \(-\frac{1}{\text{slope}}\)

slope at \((1, 2)\) = \(-\frac{1}{4}\), so

normal line slope = \(-\frac{1}{-\frac{1}{4}} = 4\)

equation:

\[ y - 2 = 4(x - 1) \]

\[ y = 4x - 2 \]

Problem 3

calculate \(\frac{dy}{dx}\)

a) \(x(\cos(y)) = y(\cos(x))\)

\[
\frac{d}{dx} (x \cos y) = \frac{d}{dx} (y \cos x)
\]

\[
x \frac{dy}{dx} \cos y + \cos y \frac{dx}{dx} = y \frac{dy}{dx} \cos x + \cos x \frac{dy}{dx}
\]

\[
x (-\sin y) \frac{dy}{dx} + \cos y = y (-\sin x) + \cos x \frac{dy}{dx}
\]

\[
(\cos x + x \sin y) \frac{dy}{dx} = \cos y + y \sin x
\]

\[
\frac{dy}{dx} = \frac{\cos y + y \sin x}{\cos x + x \sin y}
\]
Problem 3 (cont+)

b) \(e^x = \cos(x-y)\)
\[
\frac{d}{dx} (e^x) = \frac{d}{dx} (\cos(x-y)) \\
\frac{d}{dx} (e^x) = -\sin(x-y) \cdot \frac{d}{dx} (x-y) \\
e^x = -\sin(x-y) \cdot (1 - \frac{dy}{dx}) \\
e^x + \sin(x-y) = \sin(x-y) \frac{dy}{dx} \\
\frac{dy}{dx} = \frac{e^x + \sin(x-y)}{\sin(x-y)} \\
\frac{dy}{dx} = \frac{e^x}{\sin(x-y) - 1}
\]

c) \(y = \sin(xy)\)
\[
\frac{d}{dx} (y) = \frac{d}{dx} (\sin(xy)) \\
\frac{dy}{dx} = \cos(xy) \cdot \frac{d}{dx} (xy) \\
\frac{dy}{dx} = \cos(xy) \cdot (x \frac{dy}{dx} + y \frac{dx}{dx}) \\
\frac{dy}{dx} = \cos(xy) \cdot (x \frac{dy}{dx} + y) \\
\frac{dy}{dx} (1 - x \cdot \cos(xy)) = y \cos(xy) \\
\frac{dy}{dx} \frac{y \cdot \cos(xy)}{1 - x \cdot \cos(xy)}
\]