Existence of Attracting Invariant Solution Manifolds in the PDE Setting
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1 Introduction

We start with the equation

\[ \dot{u}(t) = Au(t) + R(u(t)) + M(u'), \quad (1) \]

which is the motivation for a delayed wave equation. Here \( u'(\theta) = u(t + \theta) : [-r, 0] \to X \) for \( r \) which we specify at the start.

This is a very general model for several reaction-diffusion terms with a delay in the reaction term, but we ultimately seek to apply it in this talk to the Sine-Gordon equation, which arises a model for wave propagation in quantum field theory. The abstract equation above also applies in mechanical settings, for example in viscoelastic dynamics where reaction to resisting shear flow occurs after a finite time delay.

We want to regard this as a perturbation of the following ode and apply a variation of constants formula:

\[ \dot{u}(t) = Au(t) + R(u(t)). \quad (2) \]

Here are the three assumptions we rely on:

1. \( A \) is the generator of a strongly continuous contraction group of bounded operators \( \{T(t)\}_{t \in \mathbb{R}} \) on a Banach space \( X \). Recall this means that
   (a) \( T(0) = I \),
   (b) \( T(t + s) = T(t)T(s) \),
   (c) \( \lim_{t \to 0} T(t)x = x \) for all \( x \in X \),
   (d) \( Ax := \lim_{t \to 0} \frac{T(t)x - x}{t} \) where the limit exists.

2. The (nonlinear) operator \( R : X \to X \) is locally Lipschitz continuous, Frechet differentiable, and \( R'(u) \) is also locally Lipschitz continuous in \( u \). Moreover, for any \( x \in X \) and \( g \in C([-r, \infty), X) \), the following problem

\[ \dot{u}(t) = Au(t) + R(u(t)) + g(t) \quad (3) \]

has a solution on \([-r, \infty)\) that looks like...
\[ u(t) = T(t)u(0) + \int_0^t T(t-s)(R'(u(s)) + g'(s))ds. \] (4)

In Wu, this is called the mild solution.

3. The function \( M : C = C([-r,0], X) \to X \) is also locally Lipschitz continuous.

2 The Integral Equation

For any \( x \in X \), let \( w(t,x) \) be the solution to the integral equation

\[ w(t,x) = T(t)x + \int_0^t T(t-s)R(w(s,x))ds. \] (5)

We have our first lemma:

**Lemma 1.** For each \( t \geq -r \), the solution mapping \( w(t, \cdot) : X \to X \) is \( C^1 \) and \( \Gamma(t,x) \equiv \frac{\partial w}{\partial x} \) is the mild solution of the following operator evolution equation:

\[ \frac{\partial}{\partial t} \Gamma = A\Gamma + R'(w(t,x))\Gamma(t,x), \quad t \geq -r \]
\[ \Gamma(0,x) = \text{Id} : X \to X \] (6)

**Proof:**

To show the mild solution part, just take \( \frac{\partial w}{\partial x} \) of (5).

Now for the \( C^1 \) mapping part: take \( y \in X \) and define \( \delta w(t,x,y) = w(t,x+y) - w(t,x) \). Then we want

\[ |\delta w(t,x,y) - \Gamma(t,x)y| = o(\|y\|) \]

as \( y \to 0 \). To show this, begin with the left hand side and proceed using FTC and the usual inequalities:

\[ |\delta w(t,x,y) - \Gamma(t,x)y| \leq \int_0^t |T(t-s)| |R(w(s,x+y) - R(w(s,x)) - R'(w(s,x))\Gamma(s,x)y| ds \]
\[ \leq \int_0^t \left[ \int_0^1 R'(w(s,x) + \lambda\delta w(s,x,y)) \cdot (\delta w(s,x,y) - \Gamma(s,x)y)d\lambda \right] ds \]
\[ + \int_0^t \left[ \int_0^1 [R'(w(s,x) + \lambda\delta w(s,x,y)) - R'(w(s,x))]\Gamma(s,x)y]d\lambda \right] ds. \]
Fixing a $t_1 \geq -r$ and letting $|y| \leq 1$, define

$$E_1(t_1, x, y) = \sup_{s \in [-r, t_1]} \left| \int_0^1 R'(w(s, x) + \lambda \delta(s, x, y))d\lambda \right|$$

$$E_2(t_1, x) = e^{t_1 |V|},$$

$$V = \sup_{s \in [-r, t_1]} |R'(w(s, x))|.$$

Notice that the definition of (5) gives us that

$$\sup_{-r \leq t \leq t_1} |\Gamma(t, x)| \leq E_2(t_1, x).$$

So for any $-r \leq t \leq t_1$, we have

$$|\delta w(t, x, y) - \Gamma(t, x)y| \leq E_1(t_1, x, y) \left| \int_0^t |\delta w(s, x, y) - \Gamma(s, x)y| ds \right|$$

$$+ |y| \left| \int_0^t E_2(t_1, x) \int_0^t \left| R'(w(s, x) + \lambda \delta w(s, x, y)) - R'(w(s, x)) \right| d\lambda ds \right|$$

Now, we have that $R'$ is continuous by assumption and $w$ is continuous also, so Gronwall's inequality implies what we needed. □

Recall Gronwall's inequality:

Let $u(t) \leq \beta(t)u(t)$ for $t \in I^0$. Then $u(t) \leq u(a) \exp(\int_a^t \beta(s)ds) = u$. Let $u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds$. Then $u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp[\int_s^t \beta(r)dr]ds$. Furthermore if $\alpha$ is non-decreasing, then

$$u(t) \leq \alpha(t) \exp[\int_a^t \beta(s)ds].$$

### 3 The Variation of Constants Formula, Nondelayed Equation

The method that is used to relate (1) with (2) is the following problem:

$$\dot{v}(t) = Av(t) + R(v(t)) + g(t)$$

$$v(0) = x,$$
where \( g \in C([-r, \infty), X) \) and \( x \in X \). Recall from assumption 2 that this formula has a mild solution. In fact we can take things a step further and write down the variation of constants formula:

\[
v(t) = w(t, x) + \int_0^t \Gamma(t-s, v(s)) g(s) \, ds, \tag{8}
\]

where \( t \geq -r \).

Now we consider the following undelayed equation:

\[
\dot{w}(t) = Aw(t) + R(w(t)) + K_M(w(t)), \tag{9}
\]

where we call the map \( K_M \) a perturbation, which is associated with \( M \) and will be specified in a little bit.

4 The \( K(Y) \)-property

We say the delay equation (1) satisfies the \( K(Y) \)-property for a subset \( Y \subset X \) if there is a continuous mapping \( K_M : Y \to X \) such that

\[
K_M(x) = M(\xi(\cdot, x)) \tag{10}
\]

for all \( x \in Y \), where \( \xi(\cdot, x) \) solves (9) on the segment \([-r, 0]\) through \( \xi(0) = x \).

Now we have a main theorem:

**Theorem 1.** *For any \( d > 0 \) there is an \( r_0 > 0 \) such that if \( r \in (0, r_0) \), then the delay equation satisfies the \( K(X_d) \) property for \( X_d \equiv \{|x| \leq d\} \).*

Before we continue with the proof of the main theorem, let’s discuss how we expect \( K_M \in C(K_d, X) \) to be constructed. So we want to find a \( K_M \) such that

\[
K_M(x) = M(\xi(\cdot, x))
\]

for all \( x \in X_d \).

Define

\[
p_M(t, x) = \xi(t, x) - w(t, x, R),
\]

\[
5
\]
where \( t \in [-r, 0] \) and \( w(t, x, R) \) is the solution of the nondelay equation with \( R + K_M \) replaced by just \( R \). (note this is just the unperturbed version).

Then
\[
K_M(x) = M(w(\cdot, x, R) + p_M(\cdot, x)).
\]

Finally, the variation of constant formula implies that \( p_M \) is a fixed point of the following map:
\[
G_M(p)(\theta, x) = \int_0^\theta \Gamma(\theta - s, v(s, x))M(v^s)ds,
\]

where we let
\[
v(s, x) = w(s, x, R) + p(s, x).
\]

Finally, let’s introduce the following useful set \( S_{d, \delta, \lambda} \), which consists of all bounded continuous functions \( p : [-r, 0] \times X_d \rightarrow X \) such that
1. \( p(0, x) = 0 \),
2. \( |p(t, x)| \leq \delta \),
3. \( p \) is Lipschitz in the second (spatial) entry) with constant \( \lambda \).

Couple nice things about the set \( S_{d, \delta, \lambda} \):
1. \( p \in S_{d, \delta, \lambda} \) can be extended to \([-r, 0] \times X \) by specifying \( p(t, x) = p(t, x/|x|) \) if \( |x| > d \).
2. \( S_{d, \delta, \lambda} \) is a closed, convex subset of the Banach space of bounded continuous functions from \([-r, 0] \times X_d \) to \( X \) with the supremum norm \( \|\cdot\|_\infty \).

Now we list some lemmas that are useful in proving the first theorem:

**Lemma 2.** Let \( x, x_1, x_2 \in X_d \) and furthermore specify that \( |t| \leq r \), and \( |x_1|, |x| \leq N_d + \delta \). We have:
\[
|w(t, x_1, R) - w(t, x_2, R)| \leq e^{\alpha|t|}|x_1 - x_2|,
\]
\[
|\Gamma(t, x)| \leq e^{\alpha|t|},
\]
\[
|\Gamma(t, x_1) - \Gamma(t, x_2)| \leq F(\alpha, R, t)|x_1 - x_2|,
\]

where $F(\alpha, R, t)$ is a positive scalar function not depending on $x_i$ such that $F \to \infty$ exponentially fast as $t \to \infty$ and such that if $t = 0$, then $F(\alpha \to \infty) \to 0$.

**Lemma 3.** If $p \in S_{d,\delta,\lambda}$, $t \in [-r, 0]$ and $x \in X_d$, then

$$|G_M(p)(t, x)| \leq \frac{1}{\alpha}(e^{\alpha r} - 1)[B(M) + \delta L(M)],$$

where $B(M)$ is the upper bound of $M(\psi)$ for all $\psi \in C_X$ within a $\delta$-distance of the following set $P_{d,\delta}$:

$$P_{d,\delta} = \{ w \in C_X : \|x\| \leq N_d + \delta \}$$

$$N_d = \sup_{x \in X_d : |t| \leq r} |w(t, x, R)|$$

**Lemma 4.** For $p, t, x_i$ as in the lemma above, we have

$$|G_M(p)(t, x_1) - G_M(p)(t, x_2)| \leq Q |x_1 - x_2|,$$

where $Q = Q(R, B(M), L(M), \delta)$ is a big messy ‘constant’ term.

**Lemma 5.** For $p, q \in S_{d,\delta,\lambda}$, we have

$$\|G_M(p) - G_M(q)\|_{\infty} \leq Z \|p - q\|_{\infty},$$

where $Z = Z(R, B, M, L, \lambda)$ is a similarly unappealing ‘constant’ term.

Now we can prove the $K(Y)$-property for the delay equation.

**Proof:**

By our collected lemmas, let $t \in [-r, 0]$, $x, x_1, x_2 \in X_d$, and $p, q \in S_{d,\delta,\lambda}$. Then we get

$$|G_M(p)(t, x)| \leq \delta$$

$$|G_M(p)(t, x_1) - G_M(p)(t, x_2)| \leq \lambda |x_1 - x_2|,$$

$$\|G_M(p) - G_M(q)\|_{\infty} \leq \nu \|p - q\|_{\infty},$$

where $\nu \in (0, 1)$ is a constant.

Because $G_M$ is defined as an integral from 0 to $t$, then $G_M(p)(0, x) = 0$. So $G_M$ maps $S_{d,\delta,\lambda}$ to itself and is a contraction (by above).
So it has a fixed point, \( p_M \). Then define \( K_M \) by

\[
K_M(x) = M(w(\cdot, x, R) + p_M(\cdot, x))
\]

(as before).

Now take

\[
y(t) = w(t, x, R) + p_M(t, x),
\]

where \( t \in [-r, 0] \) and \( y(t) \in \Omega_d \).

Then

\[
y(t) = w(t, x, R) + G(p_M(t, x))
\]

\[
= w(t, x, R) + \int_0^t \Gamma(t - s, y(s)) K_M(y(s)) ds.
\]

By section 3, we have that \( y(t) \) is the solution of the nondelay equation with the initial condition \( y(0) = x \). But this is what we have been calling \( \xi(t, x) \).

So

\[
\xi(t, x) = w(t, x, R) + p_M(t, x).
\]

Applying \( M \) to both sides, we obtain what we wanted. □

5 \( K(Y) \)-invariant manifolds

Now that we know that the delay equation satisfies the \( K(Y) \)-property for a small enough delay, we can proceed to define something new:

Let the delay equation satisfy the \( K(Y) \)-property. A \( C^0 \)-manifold \( M \) in \( C \) is called a \( K(Y) \)-invariant manifold for the delay equation if

1. For any \( \phi \in M \) the solution of the delay equation with initial condition \( u_0 = \phi \) exists and is in \( M \) for all \( t \in \mathbb{R} \) and

2. For any \( \phi \in M \), the solution of the delay equation with \( u_0 = \phi \) satisfies (9) for \( t \geq -r \).
Now we can state the existence of a $K(Y)$-invariant manifold for the delay equation:

**Theorem 2.** Assume $r$ is sufficiently small. Then if for some triple $(d, \delta, \lambda)$ there is a subset $D_{d,\delta,\lambda} \in X_d$ such that the mild solution $\xi(t,x)$ exists for $t \in [-r, \infty)$ and $x \in D_{d,\delta,\lambda}$, and if

\[ A \equiv \{ \xi(t,x); t \geq 0, x \in D_{d,\delta,\lambda} \in X_d \} \subset X_d, \]

then the delay equation has a $K(X_d)$-invariant manifold.

**Proof:**
The argument is quite concise. We have

\[
K_M(\xi(t,x)) = M[\xi(\cdot,\cdot),\xi(t,x))]
\]
\[
= M[w(t,\xi(\cdot,\cdot)) + R(p_M(\cdot,\cdot),\xi(\cdot,\cdot))]
\]
\[
= M[\xi(t+\cdot,\cdot)]
\]

for any $t \in [-r, \infty)$ and $x \in D_{d,\delta,\lambda}$. Then the manifold $\hat{A}$ defined by

\[
\hat{A} = \{ \xi^t(\cdot,x) \in C : t \geq 0, x \in D_{d,\delta,\lambda} \in X_d \}
\]

is an invariant manifold satisfying both conditions above, so is a $K(X_d)$-invariant manifold for the delay equation. □

This theorem can be strengthened to assert the existence of a $K(X)$-invariant manifold if there are stronger conditions on $R$ and $M$:

**Corollary 1.** Let $r$ be small. If $R'$ is uniformly bounded and uniformly Lipschitz continuous on $X$, and if $M : C \rightarrow X$ is uniformly bounded and uniformly Lipschitz continuous in the neighborhood

\[
N(M_0, \delta) = \{ \psi \in C : \text{dist}(\psi, M_0) \leq \delta \}
\]

where

\[
M_0 = \{ r(\cdot, x) \in C : x \in X \},
\]

then the delay equation has a $K$-invariant manifold.
Proof:

Define the quantities $\alpha, \lambda(R), B(M), \text{and } L(M)$ by

\[
\begin{align*}
|R'(x)| & \leq \alpha, \\
|R'(x_1) - R'(x_2)| & \leq \lambda(R)\|x_1 - x_2\|, \\
M(\psi) & \leq B(M) \\
|M(\psi_1) - M(\psi_2)| & \leq L(M)\|\psi_1 - \psi_2\|,
\end{align*}
\]

where $x, x_1, x_2 \in X, \psi, \psi_1, \psi_2 \in N(M_0, \delta)$.

Then we can define a set $S(\delta, \lambda)$ in the same way that we did for $S(d, \delta, \lambda)$, except we replace $X_d$ by $X$.

Then one can reprove lemma x but for $x, x_i \in X$.

One can reprove lemmas y-z to obtain $K_M$, and obtain $P_M$ as a fixed point of the same mapping $G_M$ as before, but now we have $x \in X$.

Now we have that $K_M$ is uniformly bounded on $X$. So for any $x \in X$, the global solution $\xi(t, x)$ exists and therefore the quality holds for any $x$ and for all $t \geq 0$.

Therefore the manifold $M$ is a $K$-invariant manifold for the delay equation since

\[ M = \{ w^t(\cdot, x, R + K_M) : t \geq 0, x \in X \}. \]

6 Smoothness and Attractivity of Invariant Manifolds

We state briefly (and without proof) two lemmas. The first is a regularity result on the smoothness of the $K$-invariant manifold:

**Lemma 6.** Let $R : X \to X$ be a $C^2$ mapping with uniformly bounded Frechet derivative up to order 2. Let $M : C \to X$ be $C^1$ having a uniformly bounded derivative on the neighborhood $N(M_0, \delta)$. Then for a suitably small time delay $r$ there is a $C^1$ manifold $M$ which is $K$-invariant for the delay equation.

The second statement describes the exponential attractivity of $M$:

**Lemma 7.** Suppose that everything holds as above. Let $r$ be sufficiently small. Assume a trajectory of the delay equation satisfies the following,

\[ \|u^t - \xi(\cdot, u(t))\| < \delta \]
for $t \in [0, h]$ for $h$ a positive constant.

Then there exist constants $\sigma > 0$ and $\eta > 0$ such that

$$\|u^t - \xi(\cdot, u(t))\| < \delta$$

for all $t \geq h$, and furthermore such that

$$\text{dist}(u^t, M) \leq \eta e^{-\sigma t}$$

for all $t \geq h$.

We can put everything together now to prove the main existence and exponential attractivity theorem:

**Theorem 3.** Let the assumptions be as in the previous lemma. For sufficiently small time delay $r > 0$, there is a $K$-invariant manifold $M$ for the delay equation such that $M$ attracts at a uniform exponential rate all the orbits entering the following neighborhood:

$$N(M, \delta) = \{ \psi \in C : \text{dist}(\psi, M) \leq \delta \}$$

Proof:

The delay equation is autonomous, so consider without loss of generality an orbit starting at $u_0$ such that

$$\text{dist}(u_0, M) \leq \delta,$$

where $\delta < \gamma$, the second constant defined as the delta in the previous lemma.

Then for an arbitrarily small $\varepsilon > 0$, we can find an $x_0 \in X$ such that

$$\|u_0 - \phi(\cdot, x, R + K_M)\| \leq \delta + \varepsilon.$$

Therefore

$$\|u_0 - \xi(\cdot, u_0(0))\| \leq \|u_0 - \xi(\cdot, x_0)\| + \|\xi(\cdot, x_0) - \xi(\cdot, x_0)\| \leq (\delta + \varepsilon)(1 + e^{ar} + \lambda),$$
where we have used one of the lemmas.

So if we pick

\[ 0 < \delta < \frac{\gamma}{1 + e^{\alpha r} + \lambda} \]

then by continuity of the solutions \( u^t \) and \( \xi(\cdot, u(t)) \) for the delay and nondelay equations, we can always find a small enough interval \([0, h]\) such that

\[ \| u_0 - \xi(\cdot, u(t)) \| < \delta \]

for \( t \in [0, h] \). Then apply lemma 3. \( \square \)

7 Example System: Sine-Gordon Equation with Delay Perturbation

We take

\[ \frac{\partial^2 u}{\partial t^2} - \Delta u + \beta \sin u = N(u_t) \]

in the space \([0, \infty) \times (a, b)\) (bounded in space), \( \beta \) a constant, \( u(t) = u(t, \cdot) \), and the following Dirichlet boundary conditions and initial condition are satisfied:

\[ u = 0 \quad \text{on} \quad [0, \infty) \times \{a, b\} \]

\[ \left( u(\theta, x), \frac{\partial u}{\partial t}(\theta, x) \right) = \omega_0(\theta, x) \quad (\theta, x) \in [-r, 0] \times [a, b]. \]

The trick is to specify the ambient (Banach) spaces and define the appropriate solution operators in such a way that assumptions 1—3 are satisfied in Section 2. This will immediately allow us to assert the existence of an invariant attracting solution manifold:

Set \( \Omega = (a, b) \) and define the following spaces:

\[ H = L^2(\Omega) \]
\[ U = H^1_0(\Omega) \]
\[ X = U \times H \]
\[ C_U = C([-r, 0], U) \]
\[ C = C([-r, 0], X). \]
We also assume that $N : C_U \rightarrow H$ is $C^1$, globally bounded and also globally Lipschitz continuous. Define the following operators:

$$ A_T = \begin{pmatrix} 0 & I \\ -\Delta u & 0 \end{pmatrix} : H^2(\Omega) \cap H^1_0(\Omega) \times U \rightarrow X $$

$$ R \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\beta \sin u \end{pmatrix}, (u,v) \in X $$

$$ M \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ N(\phi) \end{pmatrix}, (\phi,\psi) \in C. $$

Then if we call $w(t) = (u(t), \dot{u}(t))$, we can write the evolution equation as follows:

$$ \begin{align*}
\dot{w}(t) &= A_T w(t) + R(w(t)) + M(w^t), t \geq -r \\
w(0) &= \begin{pmatrix} u(0,x), \partial u / \partial t (0, x) \end{pmatrix}.
\end{align*} $$

A direct application of the exponential attractivity theorem gives us the following:

**Theorem:** (Invariant Manifold of the Sine-Gordon Equation with Delay)

For sufficiently small $r > 0$, the evolution equation of the Sine-Gordon equation with delay given by

$$ \frac{dw}{dt} = A_T(w(t) + R(w(t)) + M(w^t), t \geq 0 $$

has a $K$-invariant manifold $\mathcal{M}$ given by

$$ \mathcal{M} = \{ w(\cdot, x, R + K_M) : x \in U \times H \}, $$

where $K$ is the map from $U \times H$ to itself that arises from the $K(\mathcal{M})$ property.

Furthermore, this invariant manifold attracts at a uniform and exponential rate all orbits of the evolution equation entering a sufficiently small neighborhood of $\mathcal{M}$.

8 Further Study: State Space Decompositions in the Semiflow Setting

For completeness, we can briefly describe where invariant manifolds arise in such a way as to decompose the state space $C = C([-r,0], X)$ (for $X$ a Banach space) into linear subspaces characterized based on the behavior of solutions.
Let $A_T$ be the infinitesimal generator of a compact semigroup $\{T(t)\}_{t \geq 0}$ on $X$ and assume that for some $\lambda \in \mathbb{C}$ with $\Re \lambda = 0$, the following equation has a nontrivial solution in $y \in X$:

$$(A_T - \text{Id}\lambda)y + F(e^{\lambda}y) = 0,$$

where $F : C \to X$ is defined by $F\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta)$ for some mapping $\eta : [-r, 0] \to B(X, X)$ of bounded variation.

[The motivation for the above equation is actually a convenience that allows us to extract stability results from the spectrum of the operator $A_T$ and the resultant semigroup (note that the first part would allow us to solve for eigenvalues and eigenvectors for $A_T$).]

Then we obtain a cornucopia of results that remind us of the ODE version of the center manifold theorem:

1. $C = US \oplus CN \oplus S$, where $US$ is the unstable space, $CN$ is the center space, and $S$ is the stable space,

2. $\dim US + \dim CN < \infty$,

3. $U(t)|_{US \oplus CN}$ is extendible to all $t \in \mathbb{R}$,

4. there are positive constants $\gamma_{+}$ and $\gamma_{-}$ such that for any $\varepsilon > 0$ less than the minimum of these constants, there is a $K(\varepsilon) > 0$ such that

$$
\|U(t)\phi\| \leq K(\varepsilon)e^{(\gamma_{-}-\varepsilon)t}\|\phi\|, \ t \leq 0, \ \phi \in US,
$$
$$
\|U(t)\phi\| \leq K(\varepsilon)e^{t}\|\phi\|, \ t \in \mathbb{R}, \ \phi \in CN,
$$
$$
\|U(t)\phi\| \leq K(\varepsilon)e^{-(\gamma_{+}-\varepsilon)t}\|\phi\|, \ t \geq 0, \ \phi \in S,
$$

5. $US$ and $CN$ are invariant, $S$ positively invariant, under the solution semiflow $\{U(t)\}_{t \geq 0}$.

With a little bit more work, we can define a suitable analogue to the center manifold (as a subset of the center subspace), which we find to be invariant and (exponentially) attracting.

We conclude by remarking on the consequence of this invariance and exponential attraction: suitable infinite-dimensional PDE systems will tend, in an exponentially small amount of time, to an invariant space that, because of item 2 above, is finite-dimensional, under suitable conditions.

In this sense we have achieved the classical dynamical systems approach of dimension reduction to study asymptotic dynamics in the PDE setting.