Problem 1. Diagonalize the matrix \( M = \begin{bmatrix} 6 & -8 & -3 \\ 0 & 2 & 0 \\ 1 & -2 & 2 \end{bmatrix} \). (Recall that diagonalizing a matrix \( A \) means to find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).)

Solution.

The eigenvalues of \( M \) are 2, 3, 5. An eigenvector for 3 is \((1, 0, 1)^T\), an eigenvector for 2 is \((2, 1, 0)^T\), and an eigenvector for 5 is \((3, 0, 1)^T\). So, we can take \( D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \), \( P = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \).

Problem 2. Compute the products \( AB \) and \( BA \) where \( A \) is the matrix from problem 1 and \( B = \begin{bmatrix} 8 & -12 & -6 \\ 0 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \). Can you explain what you observe? (Hint: Diagonalize \( B \).)

Solution.

Notice that the eigenvectors we computed in the first problem are also eigenvectors for \( B \). Since \( B \) has three eigenvalues, \( B \) is diagonalizable and we can write \( B = PCP^{-1} \) for a diagonal matrix \( C \) whose diagonal entries are the eigenvalues of \( B \). Then using the fact that diagonal matrices commute, we have

\[
AB = PDP^{-1}PCP^{-1} = PDCP^{-1} = PCDP^{-1} = PCP^{-1}PDP^{-1} = BA.
\]

So, diagonalizable matrices with the same set of eigenvectors commute.

Problem 3. Solve the differential equations

\[
\frac{du}{dt} = \begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & -1 \\ 2 & -2 & 4 \end{bmatrix} u
\]

with initial conditions \( u(0) = (2, 2, 2)^T \).

First, we’ll write eigenpairs for the matrix: \((2, (0, 1, 1)^T)\), \((6, (1, 0, 1)^T)\), and \((4, (1, 1, 0)^T)\). Writing \( M = PDP^{-1} \), we have \( \frac{d}{dt}(P^{-1}u) = D(P^{-1}u) \). Solving the differential equations componentwise, we get \((P^{-1}u)_1 = C_1 \exp(2t)\), \((P^{-1}u)_2 = C_2 \exp(6t)\), and \((P^{-1}u)_3 = C_3 \exp(4t)\). Applying the initial condition (which means we set \( t \) equal to zero and set \( u(0) = \) initial condition), we get

\[
P^{-1}u(0) = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}.
\]
Therefore,
\[
P^{-1}u = \begin{bmatrix} \exp(2t) \\ \exp(6t) \\ \exp(4t) \end{bmatrix} \Rightarrow u = \begin{bmatrix} \exp(4t) + \exp(6t) \\ \exp(2t) + \exp(4t) \\ \exp(2t) + \exp(6t) \end{bmatrix}.
\]

**Problem 4.** Solve the differential equation \( y'' + 5y' + 4y = 0 \) with initial conditions \( y(0) = y'(0) = 1 \). (Rewrite the differential equation as a system of two coupled linear differential equations.)

**Solution.**
Rewrite the differential equation in this form:
\[
\frac{d}{dt}u = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} u
\]
where \( u = (y, y')^T \). An approach different from that taken in problem 3 is to 1) find the eigenvalues/eigenvectors of the matrix, 2) rewrite the initial condition as a linear combination of eigenvectors, and 3) write the solution in terms of eigenvalues and eigenvectors.

Eigenpairs for this matrix are \((-1, (1, -1)^T)\) and \((-4, (-1, 4)^T)\). We can write \( u(0) = (1, 1)^T = a(1, -1)^T + b(-1, 4)^T \), where \( a = 5/3 \) and \( b = 2/3 \). Then the solution is
\[
u = a \exp(-t)(1, -1)^T + b \exp(-4t)(-1, 4)^T,
\]
so that \( y = (5/3) \exp(-t) - (2/3) \exp(-4t) \).

**Problem 5.** Let \( A \) be an invertible matrix with eigenvalue/eigenvector pair \( (\lambda, v) \). Show that \( (1/\lambda, v) \) is an eigenvalue/eigenvector pair for \( A^{-1} \).

**Solution.**
This is shown by the following steps:
\[
Av = \lambda v \Rightarrow v = A^{-1}(\lambda v) = \lambda A^{-1}v \Rightarrow \lambda^{-1}v = A^{-1}v.
\]

**Problem 6.** Let \( A \) be a diagonalizable matrix. Show that there exists another matrix \( B \) such that \( A = B^2 \). (\( B \) is called a square root of \( A \)).

**Solution.**
Write \( A = PDP^{-1} \). Let \( D^{1/2} \) be a diagonal matrices whose entries are square roots of the entries of \( D \). Let \( B = PD^{1/2}P^{-1} \).

**Problem 7. a)** Let \( A \) be a symmetric matrix with distinct eigenvalues \( \lambda \) and \( \mu \). Show that if \( x \) and \( y \) are eigenvectors satisfying \( Ax = \lambda x \) and \( Ay = \mu y \), then \( x \) and \( y \) are orthogonal. (Hint: consider the expression \( \lambda x^T y \) and show that it is equal to \( \mu x^T y \).)

**b)** Now, assume that \( A \) is an \( n \times n \) symmetric matrix with \( n \) distinct eigenvalues. Show that the eigenvectors of \( A \) are a basis for \( \mathbb{R}^n \).

**Solution.**
We have \( \lambda x^T y = (\lambda x)^T y = (Ax)^T y = x^T A^T y = x^T Ay = x^T \mu y = \mu x^T y \). Since \( \lambda \neq \mu \), the only way this equation can hold is if \( x^T y = 0 \). Hence \( x \) and \( y \) are orthogonal.

Part a) showed that the eigenvectors of \( A \) are mutually orthogonal, and therefore are a linearly independent set of \( n \) vectors in \( n \)-dimensional space. So, the eigenvectors of \( A \) must be a basis for \( \mathbb{R}^n \).