New quiz format: On each of the remaining quizzes, there will be a problem from a worksheet.

**Problem 1.** Suppose we have bases \( \{v_1, v_2, v_3, v_4\} \) and \( \{w_1, w_2, w_3\} \) for the vector spaces \( V \) and \( W \), respectively. Suppose we are given a linear transformation \( T : V \to W \) defined by \( T(v_1) = w_1 + w_2 \), \( T(v_2) = 6w_2 + w_3 \), and \( T(v_3) = 22w_1 + 5w_2 + w_3 \), and \( T(v_4) = w_1 + w_2 + w_3 \). We are going to find the matrix \( A_T \) that represents \( T \).

a) Recall how a vector is expressed in terms of a linear combination of basis vectors: i.e., in \( \mathbb{R}^3 \) with the standard basis, we write \((1, 2, 3) = 1(1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1) \). Write a general vector \( v \) in \( V \): \( v = av_1 + bv_2 + cv_3 + dv_4 \).

b) Express \( T(v) \) as a linear combination of the given basis vectors of \( W \).

c) The matrix \( A_T \) representing \( T \) is the matrix that transforms the coordinates \((a, b, c, d)^T \) of \( v \) in the given basis for \( V \) to the coordinates of \( T(v) \) in the given basis for \( W \). Compute \( A_T \).

d) Check that \( A_T(1, 0, 0, 0)^T = (1, 1, 0)^T \), \( A_T(0, 1, 0, 0)^T = (0, 6, 1)^T \), \( A_T(0, 0, 1, 0)^T = (22, 5, 1)^T \), and \( A_T(0, 0, 0, 1)^T = (1, 1, 1)^T \).

**Solution.**

b) Since \( T \) is a linear transformation, \( T(v) = aT(v_1)+bT(v_2)+cT(v_3)+dT(v_4) \). By substituting, we get \( T(v) = a(w_1 + w_2)+b(6w_2 + w_3)+c(22w_1 + 5w_2 + w_3)+d(w_1 + w_2 + w_3) \). Grouping coefficients of each \( w_i \), this becomes \( T(v) = (a + 22c + d)w_1 +(a + 6b + 5c + d)w_2 + (b + c + d)w_3 \).

c) We computed the coefficients of \( T(v) \) in the \( \{w_1, w_2, w_3\} \) basis in the last part. So, we are looking for a matrix \( A_T \) that satisfies \( A_T(a, b, c, d)^T = (a + 22c + d, a + 6b + 5c + d, b + c + d)^T \). So \( A_T \) must be

\[
\begin{bmatrix}
1 & 0 & 22 & 1 \\
1 & 6 & 5 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}.
\]

d) Obvious now that we computed \( A_T \). But notice that \( i \)-th column of \( A_T \) is the coefficients of \( T(v_i) \) in the given basis of \( W \). So, we could have written down the matrix without doing a), b), and c).

**Problem 2.** a) Explain why \( B_1 = \{x - 1, (x - 1)^2, (x - 1)^3\} \) and \( B_2 = \{x - 1, x^2 - 1, x^3 - 1\} \) are bases for the vector space \( P = \{p(x) = a + bx + cx^2 + dx^3 : p(1) = 0\} \). [One explanation works for both. I suggest showing the dimension of the space is 3 and that the elements of these sets are linearly independent.]

b) If \( p \in P \), we can express \( p \) in terms of either basis. Find the matrix \( A \) that transforms the \( B_1 \)-coordinates of \( p \) into the \( B_2 \)-coordinates of \( p \).
Solution.

a) Take \( p \in P \). Then \( p(x) = a + bx + cx^2 + dx^3 = a(x - 1) + b(x^2 - 1) + c(x^3 - 1) \). So, \( B_2 \) spans \( P \). Since no nontrivial linear combination of the elements of \( B_2 \) is equal to zero, \( B_2 \) is a linearly independent set; hence, \( B_2 \) is a basis for \( P \), and therefore \( P \) is three dimensional. The same argument can be repeated for \( B_1 \) because \( B_2 \) is also a linearly independent set containing 3 elements.

b) Using the observation we made in problem 1d), it’s enough to write the elements of \( B_1 \) in terms of the elements of \( B_2 \):

\[
\begin{align*}
x - 1 &= x - 1 \\
(x - 1)^2 &= x^2 - 2x + 1 = (x^2 - 1) - 2(x - 1) \\
(x - 1)^3 &= x^3 - 3x^2 + 3x - 1 = (x^3 - 1) - 3(x^2 - 1) + 3(x - 1).
\end{align*}
\]

So the matrix we seek must map \((1, 0, 0)^T \mapsto (1, 0, 0)^T\), \((0, 1, 0)^T \mapsto (-2, 1, 0)^T\), and \((0, 0, 1)^T \mapsto (3, -3, 1)^T\). So, the change of basis matrix must be

\[
\begin{bmatrix}
1 & -2 & 3 \\
0 & 1 & -3 \\
0 & 0 & 1
\end{bmatrix}.
\]

Problem 3. a) Orthogonalize the basis \( \{(1, 0, 0)^T, (1, 1, 0)^T, (1, 1, 1)^T\} \) for \( \mathbb{R}^3 \) to create a new basis that still includes the vector \((1, 0, 0)^T\).

b) Use the Gram-Schmidt process to orthogonalize the basis \( \{(1, 0, 0)^T, (1, 2, 3)^T, (3, 2, 1)^T\} \).

Solution.

a) Did it in class; it’s clear that the new basis is \( \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\} \).

b) Call the vectors \( v_1, v_2, v_3 \) (in order). Name the new vectors \( u_1, u_2, u_3 \). I choose \( u_1 = v_1 \) (which is already normalized). Then

\[
v_2 - \text{proj}_{u_1} v_2 = v_2 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}
\]

and we define

\[
u_2 = \frac{v_2 - \text{proj}_{u_1} v_2}{\|v_2 - \text{proj}_{u_1} v_2\|} = \begin{bmatrix} 0 \\ 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}.
\]

Last, we consider \( v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3 \). Since \( u_2 \) is a unit vector, the projection matrix is

\[
\text{proj}_{u_2} = u_2 u_2^T = \begin{bmatrix} 2/\sqrt{13} & 2/\sqrt{13} & 3/\sqrt{13} \\ 0 & 4/13 & 6/13 \\ 0 & 6/13 & 9/13 \end{bmatrix}.
\]
Therefore
\[
v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3 = v_3 - \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} v_3 - \begin{bmatrix}
0 & 0 & 0 \\
0 & 4/13 & 6/13 \\
0 & 6/13 & 9/13
\end{bmatrix} v_3 \\
= \begin{bmatrix}
3 \\
-3 \\
1
\end{bmatrix} - \begin{bmatrix}
3 \\
0 \\
0
\end{bmatrix} - \begin{bmatrix}
0 \\
-6/13 \\
-9/13
\end{bmatrix} = \begin{bmatrix}
0 \\
-3 + 6/13 \\
1 + 9/13
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 \\
-33/13 \\
22/13
\end{bmatrix}.
\]

Note that \(\|v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3\| = \sqrt{(33/13)^2 + (22/13)^2} = \sqrt{1573/13^2} = \sqrt{121/13}\).
Then \(u_3\) is
\[
u_3 = \frac{v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3}{\|v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3\|} = \begin{bmatrix}
0 \\
-33/(13\sqrt{121/13}) \\
22/(13\sqrt{121/13})
\end{bmatrix} = \begin{bmatrix}
0 \\
-33/\sqrt{1573} \\
22/\sqrt{1573}
\end{bmatrix}.
\]

**Problem 4.** Suppose \(Q\) satisfies \(QQ^T = I\). Show that \(\det Q = \pm 1\).

**Proof.** Since \(\det Q = \det Q^T\), we have
\[
1 = \det I = \det(QQ^T) = \det Q \det Q^T = (\det Q)^2.
\]

**Problem 5.** (Problem 5.1.18) Use row operations to show that the 3 \(\times\) 3 “Vandermonde determinant” is
\[
\det \begin{bmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2
\end{bmatrix} = (b-a)(c-a)(c-b).
\]

[Hint: If \(E\) is an elimination matrix with 1s on the diagonal, what is \(\det E?\)]

**Proof.** Use \(LU\) factorization to get the \(U\) factor, since \(\det A = \det U\) for \(A = LU\):
\[
A = \begin{bmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & c-a & c^2-a^2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & 0 & c^2-a^2 - (\frac{c-a}{b-a})(b^2-a^2)
\end{bmatrix} = U.
\]

Then
\[
\det U = (b-a) \left( c^2-a^2 - \left( \frac{c-a}{b-a} \right) (b^2-a^2) \right) = (b-a)(c-a)((c+a)-(b+a))
\]
\[
= (b-a)(c-a)(c-b).
\]