1. Suppose we have \( u = (1, 0, 1) \) and \( v = (1, 0, -1) \). Find two unit vectors which are perpendicular to both \( u \) and \( v \). Pick the one, call it \( w \), such that the ordering \( u, v, w \) obeys the right hand rule.

**Answer:** \( u \) and \( v \) both lie in the \( x, z \) plane, so \((0,1,0)\) and \((0,-1,0)\) are two unit vectors which lie perpendicular to both \( u \) and \( v \). Using the right hand rule, we point our fingers in the direction of \( u \), then curl them towards \( v \), leaving the thumb pointing along the positive \( y \) axis. Thus, \( w = (0,1,0) \).

2. Describe the shape bounded by the surfaces \( x = 2, x = 4, z = -1, z = -3 \).

**Answer:** It is an infinite bar whose axis is parallel to the \( y \) axis and whose cross-sections are squares centered at the coordinates \( x = 3, z = -2 \) with edges (of length 2) parallel to the \( x \) and \( z \) axes.

3. Find bounding surfaces enclosing the unit hemisphere whose flat side is centered at \((3,1,4)\) and is parallel to and facing \( y = 0 \).

**Answer:** We start with the surface of a unit sphere centered at \((3,1,4)\). This is the set of points which satisfy the equation \((x-3)^2 + (y-1)^2 + (z-4)^2 = 1\). Now, we split the sphere in half with a plane parallel to \( y = 0 \) (the \( x, z \) plane); the plane parallel to \( y = 0 \) which goes through the center of the sphere is the plane \( y = 1 \). Observe that we now have two finite volumes enclosed by surface, so in order to get the described volume we need to specify that our surfaces only contain points where \( y \geq 1 \). This gives three equations for two bounding surfaces: the first surface is the set of points \((x,y,z)\) such that \((x-3)^2 + (y-1)^2 + (z-4)^2 = 1\) and \( y \geq 1 \), and the second surface is simply the plane \( y = 1 \).

4. How can you tell if \( v, w \in \mathbb{R}^3 \) are parallel?

**Answer:** Two nonzero vectors are parallel if they are scalar multiples of each other, i.e., there is a \( \lambda \) such that \( v = \lambda w \). Thus, \( v \) and \( w \) are parallel if the following holds: if any components of \( w \) are zero, then the corresponding components of \( v \) must be zero (and vice versa), and the ratios \( v_i/w_i \) of all the nonzero components must be equal. [Note that the idea of being parallel to a zero vector doesn’t make much sense, which is why we’re only considering nonzero vectors.]

5. Let \( \hat{i} = (1,0,0), \hat{j} = (0,1,0), \) and \( \hat{k} = (0,0,1) \) be the standard basis vectors. Show that any vector in \( \mathbb{R}^3 \) can be written as a linear combination of the vectors \( 5\hat{i}, \hat{j}, \) and \( 7\hat{i} + \hat{k} \). Recalling that a basis for \( \mathbb{R}^3 \) is a set of three vectors \( \{u, v, w\} \) such that any vector in \( \mathbb{R}^3 \) can be written as a linear combination of \( u,v, \) and \( w \), how many bases do you think there are for \( \mathbb{R}^3 \)? Why is \( \hat{i}, \hat{j}, \hat{k} \) called the standard basis?
Answer: Let \((a, b, c)\) be an arbitrary element of \(\mathbb{R}^3\). Then we can write \((a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}\). Since \(\hat{i} = (5\hat{i})/5\) and \(\hat{k} = (\hat{j} + \hat{k}) - \hat{j}\), we find
\[
(a, b, c) = a\hat{i} + b\hat{j} + c\hat{k} = a\left(5\hat{i}/5\right) + b\hat{j} + c\left((\hat{j} + \hat{k}) - \hat{j}\right)
= a\left(5\hat{i}/5\right) + (b - c)\hat{j} + c\hat{j} + c\hat{k}.
\]
This is just one example of infinitely many ways we can take linear combinations of the standard basis vectors to form new bases. The standard basis vectors are unit vectors in the coordinate directions, which is the obvious choice.

6. Let \(v\) and \(P\) be some vectors in \(\mathbb{R}^3\).

a) Find a parametrization of the line parallel to \(v\) and passing through the origin at \(t = 0\).

b) Find a parametrization of that line translated so that it passes through \(P\) at \(t = 0\).

c) Reparametrize so that the line passes through \(P\) at \(t = -1\). What point does the line pass through at \(t = 0\)?

d) Suppose \(P_0\) is any point on this line. Write \(P_0\) in terms of the parametrization found in b), i.e., let the line pass through \(P_0\) at some \(t_0\). Using this, rewrite the parametrization found in b) in terms of \(P_0\) rather than \(P\).

Answer: a) \(tv\). b) \(tv + P\). c) \((t + 1)v + P\). d) If \(P_0\) is on the line, then \(P_0 = t_0v + P\) for some \(t_0\). Thus, \(P = P_0 - t_0v\), and we can rewrite the line as \(P_0 + (t - t_0)v\).

7. Consider the lines \(r_1(t) = t(1, -1, 1)\) and \(r_2(t) = (1, 1, 0) + t(1, 2, 1)\).

a) Are there \(t_1\) and \(t_2\) such that \(r_1(t_1)\) and \(r_2(t_2)\) have the same \(x\) and \(y\) coordinates?

b) Are there \(t_1\) and \(t_2\) such that \(r_1(t_1)\) and \(r_2(t_2)\) have the same \(y\) and \(z\) coordinates?

c) Are there \(t_1\) and \(t_2\) such that \(r_1(t_1)\) and \(r_2(t_2)\) have the same \(z\) and \(x\) coordinates?

Answer: a) Yes, \(t_1 = 1/3\) and \(t_2 = -2/3\). b) Yes, \(t_1 = t_2 = -1/3\). c) No.