1. Using polar coordinates, find the average value of $x$ over the intersection of the unit circle with $x \geq 0$. Compare this to the computation in rectangular coordinates.

**Answer:** In polar coordinates, $x = r \cos \theta$, and the restriction $x \geq 0$ means $-\pi/2 \leq \theta \leq \pi/2$. Thus, the domain we’re integrating over has bounds $-\pi/2 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$. Recalling that the volume element in polar coordinates is $r \, dr \, d\theta$, we compute

$$\int_{\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} r^2 \cos \theta \, dr \, d\theta = \left( \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta \right) \left( \int_{0}^{1} r^2 \, dr \right)$$

$$= \left( \sin \theta \bigg|_{-\pi/2}^{\pi/2} \right) \left( \frac{1}{3} r^3 \bigg|_{0}^{1} \right)$$

$$= (1 - (-1)) \left( \frac{1}{3} \right)$$

$$= \frac{2}{3}.$$

Thus, recalling that the average value of a function $f$ over a domain $D$ is the integral of $f$ over $D$ divided by the area of $D$, we find that the average value of $x$ over the given set is $\frac{2}{3} \cdot \frac{1}{\pi} = \frac{2}{3\pi}$. This is the value we found in problem 4 on 16 Oct; doing the computation in polar coordinates is much faster!

2. Express the domain within the cylinder $x^2 + y^2 = 4$ where $0 \leq z \leq y$ in terms of cylindrical coordinates. Integrate $y^2$ over this region by doing an integral with respect to cylindrical coordinates.

**Answer:** Last time we found that these conditions also imply $y \geq 0$, which means $0 \leq \theta \leq \pi$. We also have $0 \leq r \leq 2$ because the projection of our domain onto the $xy$ plane is a semicircle of radius 2. Finally, $y = r \sin \theta$, so $0 \leq z \leq y$ means $0 \leq z \leq r \sin \theta$, and furthermore $y^2 = r^2 \sin^2 \theta$. Recalling that the volume element in cylindrical coordinates is $r \, d\theta \, dr \, dz$, we can write and
do the integral as follows:

\[
\int_0^\pi \int_0^2 \int_0^{r \sin \theta} r^3 \sin^2 \theta \, dz \, dr \, d\theta = \int_0^\pi \int_0^2 r^4 \sin^3 \theta \, dr \, d\theta
\]
\[
= \frac{2^5}{5} \int_0^\pi \sin^3 \theta \, d\theta
\]
\[
= \frac{2^5}{5} \frac{1}{12} \left( \cos(3\theta) - 9 \cos \theta \right) \bigg|_0^\pi
\]
\[
= \frac{2^5}{5} \frac{1}{12} (-1 - 9(-1 - 1))
\]
\[
= \frac{2^5}{5} \frac{1}{12} 16
\]
\[
= \frac{128}{15}
\]

just as we obtained last time.

3. Find the volume of the \([\text{axisymmetric}]\) region defined by rotation \(z = 1/y\) about the \(z\) axis and restricting \(1 \leq z \leq a\). What happens as \(a\) goes to infinity?

**Answer:** Since this set is symmetric about an axis, cylindrical coordinates are appropriate here. Clearly \(0 \leq \theta < 2\pi\) and \(1 \leq z \leq a\). The cross section of the given shape at height \(z\) is a disc of radius \(1/z\), which has area \(\pi/z^2\). Adding up the areas of all the discs gives the volume as \(\int_1^a \pi/z^2 \, dz = -\pi z^{-1}|_1^a = \pi (1 - 1/a)\). More formally, the volume of the given shape is

\[
\int_0^{2\pi} \int_0^a \int_0^{1/z} r \, dr \, d\theta \, dz = \left( \int_0^{2\pi} d\theta \right) \frac{1}{2} \int_0^a \frac{1}{z^2} \, dz
\]
\[
= 2\pi \frac{1}{2} (-1) z^{-1}|_1^a
\]
\[
= \pi \left( 1 - \frac{1}{a} \right).
\]

Clearly, as \(a\) goes to infinity the volume increases to \(\pi\).

4. Use spherical coordinates to compute the integral of \(\rho\) over the region \(x^2 + y^2 + z^2 \leq 4\), \(z \leq 1\), \(x \geq 0\). Draw a picture of the domain first!

**Answer:** I want to split the domain into two parts: \(0 \leq \phi \leq \phi_0\) and \(\phi_0 \leq \phi \leq \pi\), where \(\phi_0\) is the angle of declination of points on the circle \(\{(x, y, z) : x^2 + y^2 + z^2 = 4, z = 1\}\). Noting that the restriction \(x \geq 0\) is equivalent to restricting \(-\pi/2 \leq \theta \leq \pi/2\), the volume of the part of the domain which looks like a
sphere minus a cone is

\[
\int_{-\pi/2}^{\pi/2} \int_{\phi_0}^{\pi} \int_0^{\rho^2 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left( \int_{-\pi/2}^{\pi/2} d\theta \right) \left( \int_{\phi_0}^{\pi} \sin \phi \, d\phi \right) \left( \int_0^{\rho^2} \rho^2 \, d\rho \right) \\
= \pi \left( -\cos \phi \right)_{\phi_0}^{\pi} \left( \frac{\rho^3}{3} \right)_0^{\rho^2} \\
= \frac{\pi (2^3)}{3} (-1) (\cos \pi - \cos \phi_0) \\
= \frac{\pi (2^3)}{3} (-1) (-1 - \cos \phi_0) \\
= \frac{\pi (2^3)}{3} (1 + \cos \phi_0).
\]

[Note that this makes perfect sense because as \( \phi_0 \to 0 \) the volume goes to one half of \( \frac{4}{3} \pi (2^3) \), the volume of a sphere of radius 2.]

Now, one way to integrate the rest of the volume (the part with \( 0 \leq \phi \leq \phi_0 \)) is with respect to cylindrical coordinates. (We will do this first even though the directions just ask for integration with respect to spherical coordinates.) To figure out the ranges of \( z, \theta \) and \( r \), note that if we look at this part of the volume in cross section (say in the \( yz \) plane) we can use the Pythagorean theorem to figure out that \( z \) is ranging from \( r/\sqrt{3} \) to 1 and \( 0 \leq r \leq \sqrt{3} \), and of course \(-\pi/2 \leq \theta \leq \pi/2\). So the rest of the volume is

\[
\int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{3}} \int_{r/\sqrt{3}}^{1} r \, dz \, dr \, d\theta = \left( \int_{-\pi/2}^{\pi/2} d\theta \right) \left( \int_0^{\sqrt{3}} \left( \int_{r/\sqrt{3}}^{1} dz \right) r \, dr \right) \\
= \pi \int_0^{\sqrt{3}} \left( 1 - \frac{r}{\sqrt{3}} \right) r \, dr \\
= \pi \left( \frac{r^3}{2} - \frac{r^3}{3\sqrt{3}} \right)_0^{\sqrt{3}} \\
= \pi \left( \frac{3}{2} - \frac{3\sqrt{3}}{3\sqrt{5}} \right) \\
= \pi \left( \frac{3}{2} - 1 \right) \\
= \pi/2.
\]

[The volume of a cone (if you remember) is \( \frac{1}{3} \) × (area of base) × (height), and for our cone the area of the base is \( 3\pi \) and the height is 1. By restricting \( x \geq 0 \) we should be getting one half the volume of the full cone, which we do.]

Of course, the directions say to use spherical coordinates, so we will do that, too. Clearly, \( \theta \) should range from \(-\pi/2 \) to \( \pi/2 \), and we should let \( \phi \) be the independent variable as it ranges from 0 to \( \phi_0 \). So, all that is left
to do is figure out \( \rho \) as a function of \( \phi \) (since \( \rho \) is obviously independent of \( \theta \)). Now, at the flat top of the cut off sphere, \( z = 1 \). But \( z = \rho \cos \phi \), so

\[ 1 = \rho \cos \phi \implies \rho = \frac{1}{\cos \phi} \]

and we have easily obtained that the range for \( \rho \) is \( 0 \leq \rho \leq \frac{1}{\cos \phi} \) as \( \phi \) varies from 0 to \( \phi_0 \). Thus, the “cone” part of the volume is

\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{\phi_0} \int_{0}^{1/\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left( \int_{-\pi/2}^{\pi/2} d\theta \right) \left( \int_{0}^{\phi_0} \left( \int_{0}^{1/\cos \phi} \rho^2 \, d\rho \right) \sin \phi \, d\phi \right)
\]

\[
= \frac{\pi}{3} \int_{0}^{\phi_0} \frac{1}{\cos^3 \phi} \sin \phi \, d\phi
\]

\[
= \frac{\pi}{3} \int_{0}^{\phi_0} \tan \phi \sec^2 \phi \, d\phi
\]

and since the derivative of the tangent is the secant squared, we can do the substitution \( u(\phi) = \tan \phi \) to obtain

\[
\int_{0}^{\phi_0} \tan \phi \sec^2 \phi \, d\phi = \int_{u(0)}^{u(\phi_0)} u \, du
\]

\[
= \frac{1}{2} \left( u(\phi_0)^2 - u(0)^2 \right)
\]

\[
= \frac{1}{2} \tan^2 \phi_0.
\]

Since our previous considerations show that \( \tan \phi_0 = \sqrt{3} \), the previous expression is equal to \( 3/2 \), and we finally get that the volume of the “cone” part is \( \frac{\pi \cdot 3}{2} = \pi/2 \), as we should.

5. **BONUS:** What is the surface area of the region in problem 3 and what happens to the surface area as \( a \) goes to infinity? This is very strange.

**Answer:** Look up “Gabriel’s Horn” on Wikipedia (or elsewhere).