1. Describe vertical traces of \( z = 2x^2 + 5y^2 \) (vertical slices of this function for which \( x \) is constant or \( y \) is constant). Describe the level curves as well.

**Answer:** If \( x \) is fixed at say \( x = 5 \), then the relationship between \( z \) and \( y \) is given by \( z = 50 + 5y^2 \), which is the curve \((5, y, 50 + 5y^2)\) in the plane \( x = 5\). This curve is the graph of a parabola. In general, if \( x \) is fixed at \( x_0 \), then the resulting figure is the graph of a parabola in the plane \( x = x_0 \): \((x_0, y, 2x_0^2 + 5y^2)\). Similarly, if \( y \) is fixed at \( y = y_0 \), then the resulting figure is the graph of a parabola in the plane \( y = y_0 \): \((x, y_0, 2x^2 + 5y_0^2)\).

The level curves are ellipses whose axes are aligned with the \( x \) and \( y \) axes. For example, if \( z \) is fixed at \( z = 2 \), then by dividing by 2 we get \( 1 = x^2 + y^2/(2/5) \), which we can put into the standard form of the equation for an ellipse whose axes are aligned with the \( x \) and \( y \) axes: \( 1 = (x/\sqrt{2/5})^2 + (y/\sqrt{2/5})^2 \).

In general, if \( z \) is fixed at \( z = z_0 \), we would find the standard form of the ellipse to be \( 1 = (x/\sqrt{z_0/2})^2 + (y/\sqrt{z_0/5})^2 \).

2. Suppose there is a mountain whose north side is very steep and whose south side has a very shallow slope. What should its contour map look like?

**Answer:** There should be a bunch of concentric closed curves whose north sides are close and whose south sides are far apart.

3. Determine whether the following limits exist and find them if they do. Keep in mind that scalar multiples, sums, products, quotients (when defined), and compositions of continuous functions are continuous.

a) \[
\lim_{(x,y) \to (2,1)} \frac{x^2 - 5x + 6}{x^2 - x - 2}
\]

**Answer:** We can factor both polynomials: \( x^2 - 5x + 6 = (x - 2)(x - 3) \), \( x^2 - x - 2 = (x - 2)(x + 1) \). Therefore \[
\lim_{(x,y) \to (2,1)} \frac{x^2 - 5x + 6}{x^2 - x - 2} = \lim_{(x,y) \to (2,1)} \frac{x - 3}{x + 1} = \frac{1}{3}
\] since both numerator and denominator are defined and continuous at \((2,1)\) and the denominator is nonzero at \((2,1)\).

b) \[
\lim_{(x,y) \to (-1,-1)} \frac{xy^2 + 2xy + x + y^2 + 2y + 1}{xy + x + y + 1}
\]

**Answer:** We can factor the numerator and denominator as \((x + 1)(y + 1)^2\) and \((x + 1)(y + 1)\), respectively. Their ratio is then \((y + 1)\), which is defined and continuous at \((2,1)\). Therefore the limit is 0.

c) \[
\lim_{(x,y) \to (-1,-1)} \frac{xy + x + y + 1}{xy^2 + 2xy + x + y^2 + 2y + 1}
\]

**Answer:** Since the reciprocal of this function has limit zero at \((-1,-1)\) (shown in b)), the limit does not exist.
d) \[ \lim_{(x,y) \to (3,1)} \frac{2xy - 6y - x + 3}{-2xy + 6y + 4x - 12} \]

Answer: Numerator and denominator can be factored as \((2y - 1)(x - 3)\) and \((x - 3)(-2y + 4)\) respectively, so their quotient is \((2y - 1)/(-2y + 4)\). Both numerator and denominator of the simplified quotient are defined and continuous at \((3,1)\), so the limit is \(1/2\).

e) \[ \lim_{(x,y) \to (3,1)} \frac{-2xy + 6y + 4x - 12}{2xy - 6y - x + 3} \]

Answer: This function is the reciprocal of that analyzed in d), so the limit is the reciprocal of the non-zero limit found in d). Hence, the limit is \(2\).

f) \[ \lim_{(x,y) \to (1,1)} \frac{5}{e^{x-y}} \]

Answer: The exponential function is continuous and nonzero everywhere and polynomial functions in \(x\) and \(y\) are continuous everywhere, so the limit is equal to the function evaluated at \((x,y) = (1,1)\). Therefore the limit is \(5\).

g) \[ \lim_{(x,y) \to (0,0)} \ln \left( \frac{\sin x}{x} (y + 1) \right) \]

Answer: Using the squeeze theorem or geometrical considerations (length of the opposite side of the right triangle with angle \(\theta\) gets closer and closer to the arclength \(\theta\) as \(\theta\) gets small), \( \lim_{(x,y) \to (0,0)} \frac{\sin x}{x} = 1 \). Also, \( \lim_{(x,y) \to (0,0)} (y + 1) = 1 \). Since both limits are defined, the limit of their product is the product of their limits, which is \(1\). Since \(\ln\) is defined and continuous at all positive real numbers, the limit we seek is \(\ln\) evaluated at \(1\), which is \(0\).

h) \[ \lim_{(x,y) \to (0,0)} \frac{x}{\sqrt{x^2 + y^2}} \]

Answer: Clearly the limit is \(0\) if \((0,0)\) is approached along the \(y\) axis and the limit is \(1\) if \((0,0)\) is approached along the positive \(x\) axis (and also -1 if approached along the negative \(x\) axis, but we don’t need a third path), so the limit does not exist.

i) \[ \lim_{r \to 0} \cos \theta \]

Answer: This is the same as h) except expressed in polar coordinates. Cosine takes on all values between -1 and 1 on every circle of radius \(r\), no matter how small \(r\) is, so the limit cannot exist.

4. Using the definition of continuity, determine and show where the functions \(f(x,y) = x\) and \(g(x,y) = y\) are continuous.

Answer: First I will show that \(f\) is continuous everywhere. Let \((a,b) \in \mathbb{R}^2\) be an arbitrary point. Observe that \(|f(x,y) - f(a,b)| = |x - a|\), and that \(|x - a| = \sqrt{(x - a)^2} \leq \sqrt{(x - a)^2 + (y - b)^2} = \|(x,y) - (a,b)\|\). Therefore, given
any $\varepsilon > 0$, no matter how small, we can ensure that $|f(x, y) - f(a, b)| < \varepsilon$ by restricting $(x, y)$ to be within a distance $\varepsilon$ of $(a, b)$. This proves that $f$ is continuous at $(a, b)$, and since $(a, b)$ was arbitrary, we have proven that $f$ is continuous everywhere. The proof that $g$ is continuous everywhere is the same.

5. It is well known that $\lim_{x \to \infty} x^n/e^x = 0$ for all $n = 1, 2, 3, ...$. Using this, show that $\lim_{x \to 0} x \ln x = 0$.

Answer: Let $x = e^{-y}$. Then $x \ln x = e^{-y}(-y)$, and $x$ goes to zero if and only if $y$ goes to infinity. Therefore,

$$
\lim_{x \to 0} x \ln x = \lim_{y \to \infty} e^{-y}(-y) = -\lim_{y \to \infty} ye^{-y} = 0.
$$