1. Compute the cross product of $(3, -1, 2)$ and $(5, 6, 1)$.

**Answer:** We do the symbolic determinant

$$
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
3 & -1 & 2 \\
5 & 6 & 1 \\
\end{vmatrix}
= \hat{i}(-1 - 12) - \hat{j}(3 - 10) + \hat{k}(18 + 5).
$$

2. Suppose two vectors $u$ and $v$ in the $x,y$ plane are given in polar coordinates by $r_u = 2$, $\theta_u = \pi/4$ and $r_v = 3$, $\theta_v = 3\pi/4$. Find the cross-product $u \times v$ (and don’t convert to rectangular coordinates to do so!).

**Answer:** The direction of $u \times v$ is the positive $z$ direction (found using the right hand rule). The angle between $u$ and $v$ is $\pi/2$, so the magnitude of the cross product is $r_u r_v = 6$. So $u \times v = (0, 0, 6)$.

3. Show that if $u, v, w$ are ordered according to the right hand rule, then $v, w, u$ and $w, u, v$ must also be ordered according to the right hand rule.

4. Compute the area of the parallelogram with sides $u = (0,1)$, $v = (-1,1)$, and the volume of the parallelepiped with edges $u = (0,1,0)$, $v = (-1,1,0)$, and $w = (1,1,1)$. Draw pictures. Rearrange the area of the parallelogram into a rectangle and give its side lengths. Rearrange the volume of the parallelepiped into a rectangular prism and give its edge lengths.

**Answer:** The area of a parallelogram with sides $u$ and $v$ is $|u \times v|$. To see this, note that the area of such a parallelogram is the same as the area of a rectangle with side lengths $\|u\|$ and $\|v\| \sin \theta$.

The volume of a parallelepiped with edges $u, v, w$ (ordered according to the right hand rule) has volume $u \cdot (v \times w)$, or, equally as valid (by problem 3), $w \cdot (u \times v)$. Since the height of $w$ above the plane spanned by $u$ and $v$ is $\|w\| \cos \phi$, where $\phi$ is the angle between $w$ and the vector $u \times v$, the volume of the parallelepiped has the same area as a rectangular prism with edge lengths $\|u\|$, $\|v\| \sin \theta$, $\|w\| \cos \phi$.

A note about how problem 3 is used above: First of all, a parallelepiped is defined by three vectors, and the volume of the parallelepiped doesn’t change depending on what labels you put on the vectors. (What you call things doesn’t change what they are.) However, the FORMULA $u \cdot (v \times w)$ produces a positive number only if we label the vectors according to the right-hand rule (you can work out why this is). This is why $u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)$. Of course, you could just take the volume of the parallelepiped to be $|u \cdot (v \times w)|$, in which case you needn’t care whether $u, v, w$ are labelled according to the right hand rule or not.
5. Find the equation of the plane passing through the origin and perpendicular to the vector \((3, -1, 6)\).

**Answer:** This plane is the set of all vectors \((x, y, z)\) orthogonal to the vector \((3, -1, 6)\), so the plane is the set of points that satisfies \(3x - y + 6z = 0\).

6. Find the equation of the plane passing through the point \((1, 1, 1)\) and perpendicular to the vector \((3, -1, 6)\).

**Answer:** This plane is the set of all vectors \((x, y, z)\) such that \((x, y, z) - (1, 1, 1)\) is perpendicular to the vector \((3, -1, 6)\), meaning \(3(x - 1) - (y - 1) + 6(z - 1) = 0\). We can put all the constants on the right hand side thusly: \(3x - y + 6z = 3 - 1 + 6 = 8\).

7. Suppose that the equation \(5x + 2y - 2z = 1\) is given for a plane. Find a vector perpendicular to the plane and a point on the plane.

**Answer:** For an arbitrary point \((x_0, y_0, z_0)\) on the plane and a vector \((a, b, c)\) orthogonal to the plane, we can write the plane as the set of points \((x, y, z)\) such that \((x, y, z) - (x_0, y_0, z_0)\) is orthogonal to \((a, b, c)\). Using the fact that the dot product of orthogonal vectors is zero, the plane is the set of points \((x, y, z)\) such that \(a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\). If we put everything but the \(x\), \(y\) and \(z\) terms on the righthand side, we find that the plane is the set of points \((x, y, z)\) such that \(ax + by + cz = ax_0 + by_0 + cz_0\).

Using this discussion, we can see that the equation given in the problem is of the form of \(ax + by + cz = ax_0 + by_0 + cz_0\). Now we can see that our plane is perpendicular to \((5, 2, -2)\). By inspection, \((1, -1, 1)\) is a point on the plane (notice that it satisfies \(5x + 2y - 2z = 1\)).

8. We know that the planes \(3x + 4y + z + 2 = 0\) and \(3x + 4y + 4z + 5 = 0\) both contain the line \((x, -(1 + 3x)/4, -1)\).

a) What is a vector \(u\) perpendicular to the first plane?

b) What is a vector \(v\) perpendicular to the second?

c) Find the direction of the line using \(u\) and \(v\).

d) Find a point on both planes.

e) Using c) and d), find a parametrization for the line.

f) How does your parametrization compare to the one given above?

**Answer:** \(u = (3, 4, 1)\) and \(v = (3, 4, 4)\). Now, since the line lies on both planes, its direction must be perpendicular to both \(u\) and \(v\). \(u \times v = \hat{i}(16 - 4) - \hat{j}(12 - 3) + \hat{k}(12 - 12) = 12\hat{i} - 9\hat{j}\). A point that lies on the line is \((0, -1/4, -1)\). So we can write the line as \((0, -1/4, -1) + t(12, -9, 0)\). Note that this parametrization is faster than the one given above but goes in the same direction.

9. Show using dot products that given any vector \(v\) to project \(u\) onto, \(u\) is perpendicular to \(u - u\).
Answer: Since two vectors are perpendicular if and only if their dot product is zero, we should verify that $u_{||} \cdot (u - u_{||}) = 0$. Now, $u_{||} = (u \cdot v / v \cdot v) v$, so

$$u_{||} \cdot u = \left( \frac{u \cdot v}{v \cdot v} \right) (u \cdot v), \quad u_{||} \cdot u_{||} = \left( \frac{u \cdot v}{v \cdot v} \right)^2 (v \cdot v) = \frac{(u \cdot v)^2}{v \cdot v}.$$ 

It is clear from the formulas above that $u_{||} \cdot u$ and $u_{||} \cdot u_{||}$ are equal, so $u_{||} \cdot (u - u_{||}) = 0$. Therefore, $u - u_{||}$ is perpendicular to $u_{||}$. 