Discussion of "Extreme Events: Mechanisms and Prediction" by M. Farazmand and T. P. Sapsis (ASME, Applied Mechanics Reviews)

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Abstract

Farazmand and Sapsis provide a thorough review on dynamical systems and offer strategies to predict the occurrence of extreme events. The aim of this discussion is to supplement the review by presenting additional mechanisms that yield extreme events and suggesting statistical tools to assess the quality of our prediction. The perspective taken here appeals to probabilistic and statistical approaches used in modeling heavy-tailed phenomena.

1 Introduction

Extreme events are of great interest in engineering and natural sciences since they can have severe consequences. The occurrence and intensity of these events are difficult to predict since they are generated by complex physical mechanics which, usually, are partially understood and random. The authors present a most useful review on the characterization and prediction of extreme events which include mechanisms which trigger extreme events, real-time prediction, mitigation, and statistical methods to characterize such events. The focus of their study is on the mechanisms of extreme events which are illustrated by interesting examples and the construction of indicators which signal upcoming extreme events.

We briefly discuss two aspects of extreme events to supplement the review in [3]. The first is on extremes of states of simple dynamical systems caused by random inputs with heavy tails. The second is on the use of tools from multivariate extreme value distribution theory (MEVT) to quantify the quality of indicators of extreme events. This discussion aims to build on the contributions by the review paper.

2 Heavy-tail inputs

The review by Farazmand and Sapsis presents two types of physical systems. In the first type, the states satisfy deterministic dynamical systems or partial differential equations which exhibit extreme events. The second type of system is characterized by stochastic differential equations with random noise whose states experience transitions caused by the noise intensity.

We show that in addition to the behavior described in the review article, physical systems of the second type also produce extreme events aside from noise-induced transitions, e.g. dynamical systems subjected to heavy-tailed random inputs. The states of these systems take large infrequent values irrespective of the structure of the defining equation. We illustrate this statement by the simplest linear system subjected to α-stable Lévy noise.

Let $X(t)$, $t \geq 0$, be a real-valued stochastic process defined by the differential equation

$$dX(t) = -\lambda\, X(t)\, dt + dL_\alpha(t), \quad \lambda > 0, \quad t \geq 0, \tag{1}$$

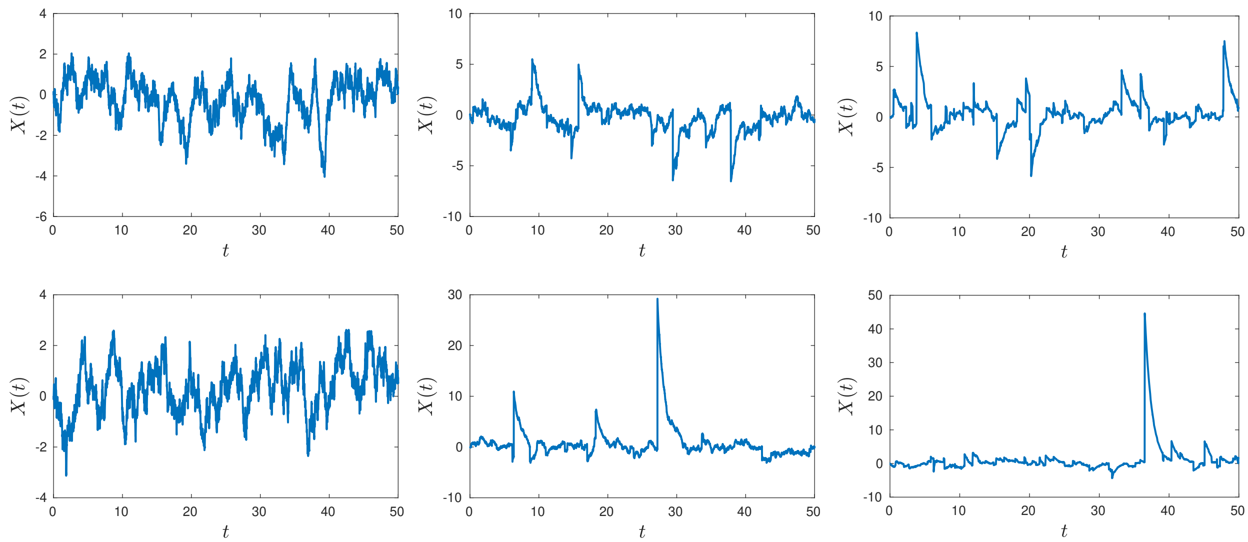
where L_α, $0 < \alpha \leq 2$, is an α-stable process. This process is continuous in probability and has independent stationary increments $dL_\alpha(t) = L_\alpha(t+dt) - L_\alpha(t)$ which are independent from the past [5] (Sect. 3.14). The characteristic function of the random variable $L_\alpha(t)$ process is $\varphi(u) = E\big[\exp(i\,u\,L_\alpha(t))\big] = \exp\big(-t\,|u|^\alpha\big)$ since it has independent stationary increments and $E\big[\exp(i\,u\,dL_\alpha(t))\big] = \exp\big(-dt\,|u|^\alpha\big)$ [7] (Chap. 3). For $\alpha = 2$, $L_\alpha(t)$ is a scaled Brownian motion since $\varphi(u) = \exp\big(-t\,u^2\big)$ so that it is Gaussian with mean zero and variance $2\,t$.

The solution of Eq. 1 for $X(0) = 0$ is the stochastic integral

$$X(t) = \int_0^t e^{-\lambda\,(t-s)}\, dL_\alpha(s), \quad t \geq 0, \tag{2}$$

which is an α-stable process. Samples of $X(t)$ can be generated from the finite difference version $X(t+\Delta t) = \big(1 - \lambda\,\Delta t\big)\, X(t) + \Delta L_\alpha(t)$ of Eq. 1, where $\Delta t > 0$ and $\Delta L_\alpha(t) = L_\alpha(t+\Delta t) - L_\alpha(t)$. The top and bottom panels of Figure 1 show two sets of samples of $X(t)$, $\lambda = 1$, for $\alpha = 2, 1.5$ and 1 (left, middle, and right panels). Figure 2 shows 50000 samples of the random variable $\max_{0 \leq t \leq 50} |X(t)|$ fitted to a generalized extreme value distribution (GEV) with the corresponding shape ξ, scale σ, and location μ parameters [6]. The extreme event in this example pertains to bursts in the state trajectories which are characterized by large magnitudes. Observe that extreme events are present only for $\alpha < 2$ since the tail of the driving noise is heavier that that of the Gaussian noise. The magnitude of the extreme events are larger for smaller values of α. Note that the scale of the samples of $X(t)$ in Figure 1 and the range of $\max_{0 \leq t \leq 50} |X(t)|$ in Figure 2 differ across the panels.

The example above shows that simple dynamical systems driven by certain types of noise may exhibit bursts in sample paths of the state which may be considered extreme. In this case, the extreme event is caused by the random noise driving the system. This distinguishes it from the deterministic dynamical systems in the review paper [3]. There, the *randomness* in the state, and therefore the occurrence of extreme events, is caused solely by the dynamics. In [3], it seems that the invariant measure defined on the state space



(a) $\alpha = 2$ (b) $\alpha = 1.5$ (c) $\alpha = 1$

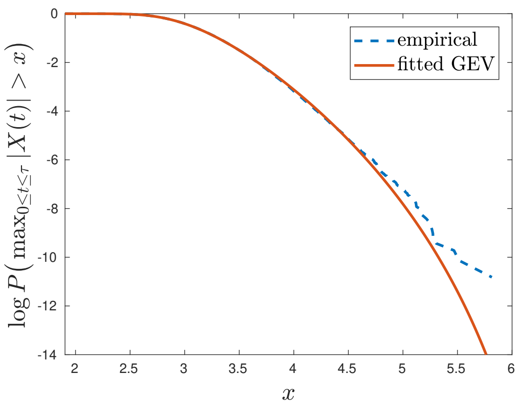
Figure 1: Samples of $X(t)$ for $\alpha = 2$, 1.5, and 1.0 (left, middle, and right panels)

is derived from long-time simulations under the assumptions that the state trajectory is modeled as an ergodic, stationary stochastic process. Samples of the state are therefore obtained by concatenating long-time simulations of the system initiated at different initial conditions [3, Section 4.2]. In this respect, we believe that the authors account in their simulations for possible pitfalls caused by numerically simulating deterministic systems with chaotic dynamics [1, 2, 8] and that the observed chaotic behavior is not due to numerical errors [9].

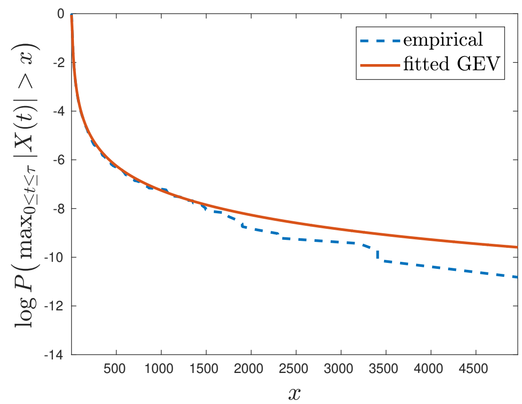
3 Indicators of extreme events

Indicators of extreme events of the type proposed in [3] are functionals of system states which are simpler than quantities of interest and signal upcoming extreme events. Conditional probabilities are used to quantify and assess the indicator performance via the likelihoods of correct predictions/rejections and of false negatives/positives.

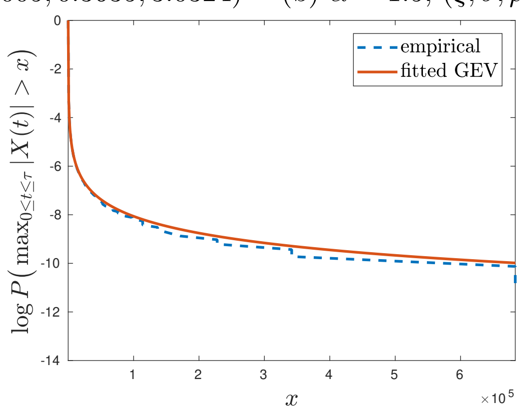
Alternative tools, referred to as spectral measures, are provided by the multivariate extreme value theory (MEVD). They quantify the likelihood that two or more components of a random vector are simultaneously large [6] (Chap. 6, 9). We give a heuristic description of these metrics. For simplicity, consider a two dimensional random vector $X = (X_1, X_2)$, $X_1, X_2 > 0$ whose components have the same distribution. The polar representation of this vector is $X = (X_1, X_2) = \big(V \cos(\Theta), V \sin(\Theta)\big)$, where $V = \|X\|$ and $\Theta = \tan^{-1}(X_2/X_1)$. The polar representation of the samples $\{x^{(k)}\}$ of X is $x^{(k)} = (x_1^{(k)}, x_2^{(k)}) = \big(v^{(k)} \cos(\theta^{(k)}), v^{(k)} \sin(\theta^{(k)})\big)$, $\quad k = 1, \ldots, n$. Select $v_0 > 0$ relatively large. Samples of X with distance to origin $v^{(k)} > v_0$ are of interest. The selection of v_0 is critical to assure that the right tails of the components of X are accurately represented. Let $\{x^{(k_1)}, \ldots, x^{(k_m)}\}$ denote the subset of $\{x^{(1)}, \ldots, x^{(n)}\}$, $m \ll n$, such that $v^{(k_j)} > v_0$, $j = 1, \ldots, m$. The histogram $h(\theta)$ of $\{\theta^{(k_1)}, \ldots, \theta^{(k_m)}\}$ with support $[0, \pi/2]$ characterizes the



(a) $\alpha = 2$, $(\xi, \sigma, \mu) = (-0.1005, 0.3635, 3.0324)$



(b) $\alpha = 1.5$, $(\xi, \sigma, \mu) = (0.6869, 4.6996, 7.6860)$



(c) $\alpha = 1$, $(\xi, \sigma, \mu) = (1.0003, 31.4690, 31.9585)$

Figure 2: Empirical and fitted GEV CDF of $\max_{0 \le t \le 50} |X(t)|$ for $\alpha = 2$, 1.5, and 1.0 (upper left, upper right, and bottom panels)

dependence between X_1 and X_2 for samples of X with norm exceeding v_0. If most of the mass of $h(\theta)$ is concentrated at $\theta = 0$ and $\pi/2$, X_1 and X_2 are unlikely to be simultaneously large; their extremes are nearly independent. If most of the mass of $h(\theta)$ is concentrated in a small interval away from $\{0\}$ and $\{\pi/2\}$, X_1 and X_2 are simultaneously large with high probability; their extremes are strongly dependent. Histograms $h(\theta)$ between these limit cases describe various degrees of dependence between extremes of X_1 and X_2. This means that X_1 is a good indicator for $X(2)$ if $h(\theta)$ is concentrated in a small interval away from $\{0\}$ and $\{\pi/2\}$.

For example, let $X_i = \lambda G_0^2 + (1 - \lambda) G_i^2$, $i = 1, 2$, where G_0, G_1, G_2 are independent standard Gaussian variables. The left, middle, and right top panels of Figure 3 show scatter plots of the samples of $X = \big(X_1, X_2\big)$ for $\lambda = 0.9$, $\lambda = 0.5$, $\lambda = 0.1$ which are used to construct the spectral measures shown in the bottom panels. The spectral measure quantifies the intuition that the likelihood that X_1 and X_2 are simultaneously large decreases with λ.

We use spectral measures to assess the performance of intensity measures, which are indicators used in earthquake engineering to predict response maxima of complex dynamic systems subjected to random ground motions $A(t)$. Intensity measures are based on the heuristic hypothesis that these response maxima correlate with response maxima of simple

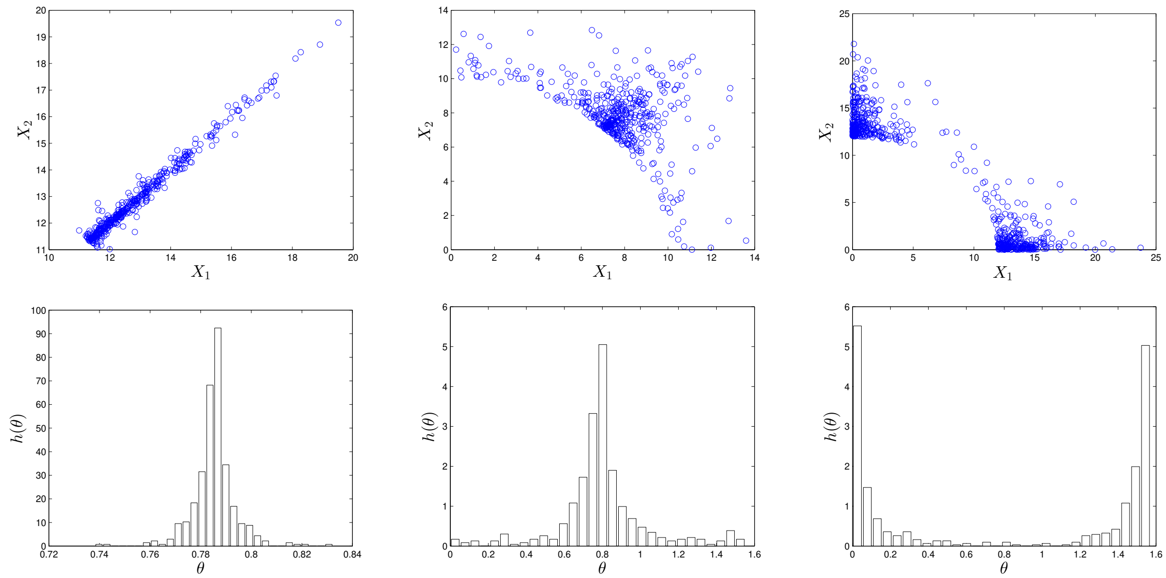


Figure 3: Large samples of (X_1, X_2) (top panels) and corresponding spectral measure (bottom panels) for $\lambda = 0.9$, $\lambda = 0.5$, $\lambda = 0.1$ (left, middle, and right panels)

linear oscillators subjected to the same input $A(t)$. Unlike the indicators derived in [3], intensity measures are not based on physical and mathematical arguments.

Suppose that the quantity of interest is the random variable $D = \max_{0 \le t \le \tau} |X(t)|$, where $X(t)$ is defined by

$$\ddot{X}(t) + 2\,\zeta\,\nu_0\,\dot{X}(t) + \nu_0^2\,\big(\rho\,X(t) + (1-\rho)\,W(t)\big) = -A(t),$$

where $\dot{W}(t) = \gamma\,\dot{X}(t) - \alpha\,|\dot{X}(t)|\,|W(t)|^{\chi-1}\,W(t) - \beta\,\dot{X}(t)\,|W(t)|^{\chi}$, $\nu_0 = 2\,\pi$, $T = 1$, $\zeta = 0.05$, $\alpha = 0.5$, $\beta = 5$, $\gamma = 3$, $\rho = 0.1$, $\chi = 1$, and $A(t)$ is a zero-mean stationary Gaussian process with spectral density in the left panel of Figure 4. The intensity measure is $S_{\mathrm{a}}(T) = (2\,\pi/T)^2\,\max_{0 \le t \le \tau} |X_{\mathrm{sdof}}(t)|$, where $X_{\mathrm{sdof}}(t)$ is $X(t)$ for $\rho = 1$. The middle of Figure 4 shows a scatter plot of the bivariate vector $\big(S_{\mathrm{a}}(T), D\big)$. The circles mark the samples used to construct the spectral measure in the right panel of the figure. This measure shows that $S_{\mathrm{a}}(T)$ is an inadequate indicator for D even for this simple nonlinear system. For additional considerations on this metric, see [4].

4 Conclusions

We presented additional mechanisms to generate extreme events in dynamical systems which are caused by inputs with heavy-tailed distributions. It is shown that even the simplest linear dynamic system exhibit such events in heavy-tailed random environment.

The second part of the discussion relates to a metric to assess the usefulness of indicators of extreme events. It was shown that intensity measures used currently in earthquake engineering provide limited if any information on the extremes of nonlinear dynamic systems.

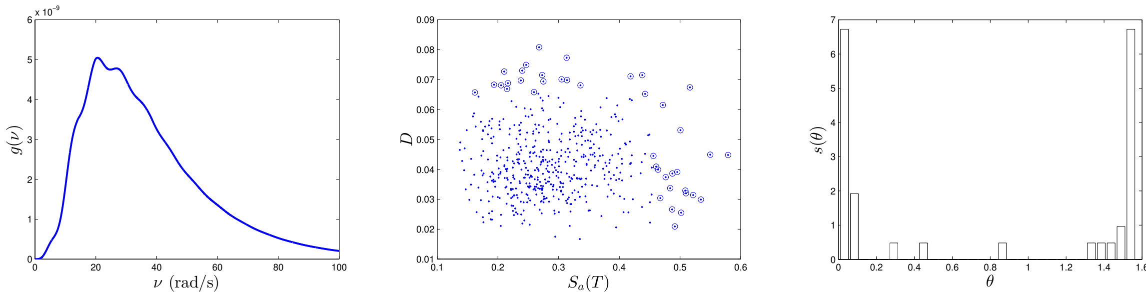


Figure 4: Spectral density of $A(t)$, scatter plot of $\big(S_{\mathrm{a}}(T), D\big)$, and spectral measure of this vector (left, middle, and right panels)

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