Volume Integrals Supplement

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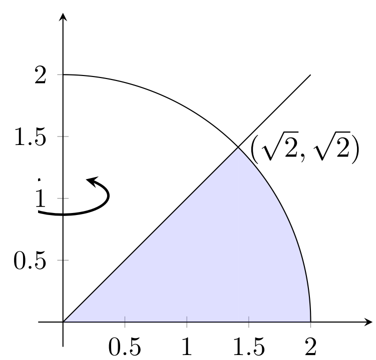
Compute the volume of the the solid of revolution obtained by revolving the region that is defined by the following set of inequalities:

$$y \leq \sqrt{4 - x^2}, \quad y \leq x, \quad \text{and} \quad y \geq 0.$$

about the y-axis.

Proof. We will be computing the volume of this solid via two different methods: the disk/washer method and the cylindrical shell method. Regardless of the method, the relevant region that is revolved to form the solid is depicted below:

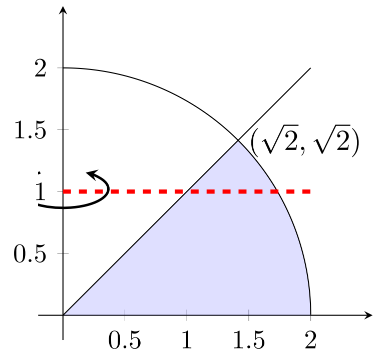


The Disk/Washer Method

If we choose to use the disk/washer method to compute the volume of this solid, we will compute the volume by computing the volumetric contributions of cross-sections that lay orthogonal to the axis of revolution. Since the axis is vertical, our cross-sections are horizontal in profile. Further, since the region does not have the axis of revolution as its boundary, these orthogonal cross-sections are washers. Third, the washer cross-sections lie at constant y-coordinates, the definite integral we construct will be respect to y. Hence, the volume contributed by each cross section is given by the expression:

$$dV = A(y)\,dy = \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$

where R and r are functions of y that describe the outer and inner radii of each washer, respectively. This gives us the following guiding diagram to help construct our definite integral:



The washer cross-sections also range across the entirety of the region in profile with a consistent pair of boundaries: the inner radius is defined by the line $y = x$ and the outer radius is defined by the curve $y = \sqrt{4 - x^2}$. We also observe that the region extends between $y = 0$ to $y = \sqrt{2}$. Therefore, we find that the definite integral representing volume is given by:

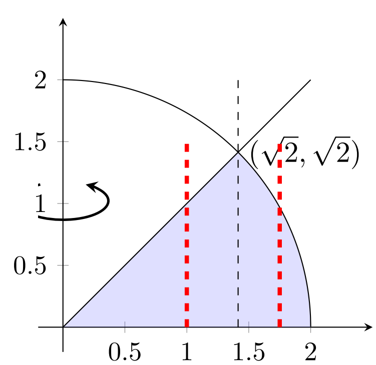
$$V = \int_0^{\sqrt{2}} A(y)\, dy = \int_0^{\sqrt{2}} \pi \left(\left[\sqrt{4 - y^2} \right]^2 - [y]^2 \right)\, dy = \pi \int_0^{\sqrt{2}} \left(4 - 2y^2\right)\, dy.$$

We can compute this definite integral to determine the volume of the solid:

$$\begin{aligned} V &= \pi \int_0^{\sqrt{2}} \left(4 - 2y^2\right)\, dy, \\ &= \pi \left(4y - \frac{2}{3} y^3 \right) \Big|_0^{\sqrt{2}}, \\ &= \frac{8\pi\sqrt{2}}{3}. \end{aligned}$$

The Cylindrical Shell Method

If we choose the cylindrical shell method to compute the volume of this solid, we will compute the volume by computing the volumetric contributions of cross-sections that in profile lie parallel to the axis of revolution. We use the following guiding diagram to guide our construction:



Since the axis is vertical, our cross-sections are vertical in profile at constant x-coordinates. Hence our definite integral we construct will be with respect to x. Also note that while the region extends between $x = 0$ and $x = 2$, cross-sections across the region are not defined by a consistent pair of boundaries. Cross-sections left of $x = \sqrt{2}$ extend from the x-axis to the line $y = x$ while cross-sections to the right extend to the curve $y = \sqrt{4 - x^2}$. Therefore, we require two definite integrals in order to represent the volume of the solid. Cross-sections to the left of $x = \sqrt{2}$ have volumetric contributions:

$$dV = 2\pi r(x)h(x)\,dx = 2\pi \cdot (x) \cdot (x - 0)\ dx = 2\pi x^2\,dx.$$

Cross-sections to the right of $x = \sqrt{2}$ have volumetric contributions:

$$dV = 2\pi r(x)h(x)\,dx = 2\pi \cdot (x) \cdot \left(\sqrt{4 - x^2} - 0\right)\ dx = 2\pi x\sqrt{4 - x^2}\,dx$$

Hence our total volume for the solid of revolution is given by:

$$\begin{aligned} V &= \int_0^{\sqrt{2}} 2\pi x^2\,dx + \int_{\sqrt{2}}^{2} 2\pi x\sqrt{4 - x^2}\,dx, \\ &= 2\pi\left(\frac{x^3}{3}\right)\Bigg|_0^{\sqrt{2}} + 2\pi\left(-\frac{2}{3}(4 - x^2)^{3/2}\right)\Bigg|_{\sqrt{2}}^{2}, \\ &= \left(\frac{4\pi\sqrt{2}}{3}\right) + \left(\frac{4\pi\sqrt{2}}{3}\right), \\ &= \frac{8\pi\sqrt{2}}{3} \end{aligned}$$

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