ESMI Applied Math - Summer 2019 | 25 Jun 2019

The Mass-Spring-Damper Model

Expected Skills.
Students can...

- *encode ODE models in MATLAB.*
- *generate trajectories and phase plots of ODE models in MATLAB.*
- *identify and classify invariant features of ODE systems.*
- *perform parameter studies of ODE models in MATLAB.*
- *contextualize mathematical statements to conclusions about the model.*

Guiding Questions

(a) How can you turn a differential equation into a system of ODEs?

(b) What is the general workflow of implementing an ODE system in MATLAB?

(c) What are some useful built-in functions of MATLAB to generate time simulations?

Encoding the Mass-Spring-Damper Model

The Mass-Spring-Damper model is one of the most common models used by engineers to model kinematic systems. From human tissue to bridges, this straightforward model features three mechanisms and can be summarized as the following second-order differential equation:

$$m\ddot{x} + c\dot{x} + kx = g(x, \dot{x}, t).$$

Here x represents the displacement of the object with mass m away from its resting position. The mass is subject to some spring force characterized by spring constant k and a damping force that resists change in motion with damping coefficient c. The function g here can be thought of as some input to the system that could depend on position, velocity, or time. For our discussion today, we will set $g = 0$.

- Turn this second-order differential equation into a system of first-order ODEs. *Hint: Set $y = \dot{x}$ and solve for a system of x and y.*
- Make some predictions about the impact of how each parameter m, c, and k impact the system state $u = \begin{bmatrix} x \\ y \end{bmatrix}$.
- Encode the mass-spring-damper model: `du = msd(t,u,p)`
- You may want to define several `myOdeFun = @(t,u) msd(t,u,p)` with p defined differently in each instance.

Analyzing the Mass-Spring-Damper Model

One application of the mass-spring-damper model is in the use of designing driver seats in vehicles. A well-engineered driver seat should be able to limit the amount of vertical displacement and velocity a driver experiences as they are traversing various road conditions like speed bumps. Let $m = c = k = 1$.

(a) Plot the solution against time with initial condition: $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Explain your figure.

(b) Plot the solution as a trajectory in the phase space diagram for this system. What do you notice?

(c) Determine any equilibrium points of this system. Which of these equilibrium points appear to be attracting? repelling?

(d) Investigate the impact of the parameters m, c, and k by varying their values and checking the resulting solutions you obtain. Make sure you investigate when these values are zero when appropriate. Write up your findings a few paragraphs.

(e) Suppose we are designing a seat to withstand periodic vibrations from the road. Set $g(t) = 10\sin(3t)$. Determine an appropriate damping coefficient and spring constant so that the driver ($m = 1$) experiences less than 0.5 displacement or velocity from rest. Try to minimize both as much as possible and document your thought process behind your design solution.

Project Ideas

The Mass-Spring-Damper model is a fantastic starting place for many projects involving kinematics. Some possible directions:

- Multiple masses-springs-dampers composed together
- Forced mass-spring-dampers studying the impact of g on the system
- Nonlinear spring term (see: Duffing equation)
- Mass-spring-damper model in 2-D (see: cantilever beam)