ESMI Applied Math - Summer 2019 14 Jun 2019

Error Analysis

Expected Skills.
Students can...

- *identify and contrast the three sources of error with floating point arithmetic.*

Discussion Questions

(a) What is the value of e?

```
e = 1 - 3*(4/3 - 1)
```

What is happening here?

(b) What is the value of e?

```
e = sqrt(1e-16 + 1) - 1
```

What is happening here?

(c) How does truncation error play into our approximation of pi using the series expansion for arctangent?

Approximating π (Again)

Recall the series expansion for arctangent:

$$\arctan(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2j+1}.$$

(a) Obtain a series expansion for π by substituting $x = \frac{1}{\sqrt{3}}$.

(b) Copy and modify your MATLAB program to approximate π now with $x = \frac{1}{\sqrt{3}}$ for $N = 11, 101, 1001, 10001, 100001$ terms.

(c) Compute the error in the approximations using the built-in constant for π.

(d) Plot the absolute error as a function of N for both of your approximations: $x = 1$ and $x = \frac{1}{\sqrt{3}}$. Explain your plot.

(e) Why does the series with center at $x = \frac{1}{\sqrt{3}}$ give a better approximation to π than one with center at $x = 1$? *Hint: Taylor's Remainder Theorem.*

Quadratic Formula

In solving for the roots of the quadratic equation $ax^2 + bx + c = 0$ with the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

there can be loss of significance when using floats.

(a) When and where does this loss of significance occur?

(b) Suggest a method to mitigate this difficulty and explain why your alternate formula avoids the pitfall that the standard quadratic formula encounters.

Two Trigonometric Functions

Consider the two functions $f(x) = 1 - \tan x$ and $g(x) = \dfrac{\cos 2x}{\cos^2 x(1 + \tan x)}$.

(a) Show that $f = g$.

(b) Which function should be used when evaluating x near $\pi/4$ and $5\pi/4$? Why?

(c) Which function should be used when evaluating x near $3\pi/4$ and $7\pi/4$? Why?

Antiderivatives

Suppose we wish to evaluate the integral

$$I(N) = \int_N^{N+1} \frac{1}{1+x^2}\, dx$$

using the antiderivatives

$$I(N) = \arctan(N+1) - \arctan(N).$$

(a) Explain why the above equation is numerically unacceptable when $N = 10^8$.

(b) Find some exact way to rewrite the above equation that is good for computation for large N. Explain why your formula is better.

Inverse Hyperbolic Sine Function

The inverse hyperbolic sine function (asinh) has a Taylor expansion given by

$$\text{asinh}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}.$$

(a) Explain why the following code to compute the $n = 1000$ Taylor coefficient returns NaN:

```
n=1000;
taylorCoefficient = (-1)^n*factorial(2*n)/(4^n*factorial(n)^2*(2*n+1))
```

(b) Write code for `taylorCoefficient` that allows you to compute $n = 1000$ accurately. What is the value?