Infinitely divisible distributions and the Lévy-Khintchine formula

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Some definitions

Let X be a real-valued random variable with law μ_X. Recall that X is said to be infinitely divisible if for every $n \in \mathbb{N}$ we can find i.i.d. random variables $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that

$$X \overset{\mathrm{d}}{=} Y_1^{(n)} + \cdots + Y_n^{(n)},$$

or equivalently if there exists a probability measure μ_Y (depending on n) such that

$$\mu_X = \underbrace{\mu_Y * \cdots * \mu_Y}_{n \text{ times}} := (\mu_Y)^{n*}.$$

Some examples are

- the Gaussian distribution.
- the compound Poisson distribution.
- the Dirac delta distribution.
- stable distributions.

Some definitions

A real-valued stochastic process $\{X_t\}_{t \geq 0}$ is said to be a Lévy process if:

1. $X_0 = 0$ a.s.
2. it has stationary and independent increments.
3. it is *stochastically continuous*, i.e. for any $a > 0$,

$$\lim_{t \to s} P(|X_t - X_s| > a) = 0.$$

Every Lévy process has a version which is càdlàg. Some examples of Lévy processes are:

- if $X_t \sim N(0, t)$ then we have a standard Brownian motion.
- if $X_t = \sum_{k=1}^{N_t} \xi_k$, where $N_t \sim \text{Poiss}(\lambda t)$ and $\{\xi_k\}_{k \geq 1}$ are i.i.d. then we have a compound Poisson process (time-changed random walk).
- first passage times for standard Brownian motion with drift.

Infinite divisibility and Lévy processes

There is a bijection between the class of Lévy processes and the class of infinitely divisible distributions.

$$\text{Lévy processes} \longleftrightarrow \text{infinitely divisible distributions}$$

- For any $t \geq 0$ and any $n \in \mathbb{N}$,

$$X_t = \left(X_{\frac{tn}{n}} - X_{\frac{(t-1)n}{n}}\right) + \cdots + (X_{\frac{1}{n}} - X_0).$$

- Furthermore, setting $\mu = \mathscr{L}(X_1)$, we have $\mathscr{L}(X_t) = \mu^{t*}$ (!).

Therefore we can either talk of a particular Lévy process or of an infinitely divisible distribution, say μ.

Lévy-Khintchine formula

The main subject of this talk is the beautiful and fundamental,

Theorem (Lévy,Khintchine)

Let μ be an infinitely divisible distribution supported on $\mathbb{R}$. Then for any $\theta \in \mathbb{R}$ its characteristic function is of the form,

$$\widehat{\mu}(\theta) = \exp\left[ia\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x 1_{|x|<1}\right) \nu(dx)\right],$$

where $a, \sigma \in \mathbb{R}$ and ν is a measure satisfying,

$$\nu(\{0\}) = 0 \qquad \textit{and} \qquad \int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty.$$

We call (a, σ, ν) the characteristic triplet of μ, and ν is referred to as the Lévy measure (or sometimes the jump measure).

Comments

- The quantity in the square brackets is called the characteristic exponent, $\psi(\theta)$.
- The cutoff function can be modified from $1_{|x|<1}$ but the drift term has to be appropriately modified.
- Notice that,

$$\left| \int_{|x|<1} (e^{i\theta x} - 1 - i\theta x)\nu(dx) \right| \leq Cu \int_{|x|<1} x^2 \nu(dx) < \infty,$$

and also,

$$\left| \int_{|x|\geq 1} (e^{i\theta x} - 1)\nu(dx) \right| \leq 2 \int_{|x|\geq 1} \nu(dx) < \infty.$$

So the integrability conditions of the Lévy measure are quite natural.

Examples of characteristic exponents

- For $\mu \sim \mathcal{N}(a, \sigma^2)$ (standard Brownian motion with drift) then,

$$\psi(\theta) = ia\theta - \frac{1}{2}\sigma^2\theta^2.$$

  Clearly $\nu = 0$.

- If $\mu \sim \mathrm{Poiss}(\lambda)$ for $\lambda > 0$ (Poisson process) then,

$$\psi(\theta) = \lambda(e^{i\theta} - 1).$$

  Notice that $\nu = \lambda\delta_1$.

- When $\mu \sim \mathrm{ComPoiss}(\lambda, F)$ we get

$$\psi(\theta) = \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1)F(dx).$$

  Observe that $a = -\lambda \int_{|x|<1} xF(dx)$ while $\nu(dx) = \lambda F(dx)$.

Lévy-Khintchine re-visited

One way to re-write the characeristic exponent is as follows,

$$\psi(\theta) = -\left[\frac{1}{2}\sigma^2\theta^2\right] + \left[\int_{\mathbb{R}}(e^{i\theta x} - 1)\nu(dx)\right] + \left[i\theta\left(a - \int_{|x|<1} x\nu(dx)\right)\right].$$

Another way to re-write the exponent has a more probabilistic interpretation, and is related to the Lévy-Itō decomposition,

$$\psi(\theta) = \left[i\theta a - \frac{1}{2}\sigma^2\theta^2\right] + \left[\int_{|x|\geq 1}(e^{i\theta x} - 1)\nu(dx)\right] + \left[\int_{|x|<1}(e^{i\theta x} - 1 - i\theta x)\nu(dx)\right].$$

The proof of this decomposition gives deeper insight into the origins of the Lévy measure.

Origin of the Lévy measure

Outline of the proof:

- First, for any Borel set, A, bounded away from zero one can define the counting process associated to the jump process of a Lévy process, counting jumps up to time t taking values in A (recall càdlàg).
- The resulting process is a Poisson process and gives rise to a Poisson random measure.
- The Lévy measure is the intensity measure of the Poisson random measure, when $A = \mathbb{R} \setminus (-1, 1)$.
- Therefore the next step is to subtract, from any given Lévy process, the portion corresponding to jumps of size ≥ 1.
- Centering this process gives rise to an L^2-martingale, which can be shown to split into the sum of a Brownian motion and a pure jump process.
- The square summability of the L^2 pure jump martingale together with the choice of A give the integrability conditions of ν.

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