MA 732
Study Guide

Autonomous Equations

Let our general autonomous equation be represented by

$$(\star) \quad u' = f(u), \;\; u(0) = z \in \Lambda, \;\; t \geq 0$$

and we will assume invariance ($z \in \Lambda \implies u(t) \in \Lambda \forall t \geq 0$) and that solutions are unique to the right.

Semi-groups

We notate solutions to $(\star)$ as $S(t)z$ $t \geq 0$ with $S(0)z = z$. We call S a semigroup or "semi-dynamical system" (or trajectory). S has the following properties

- $\frac{d}{dt}S(t)z = f\left(S(t)z\right), \, t \geq 0$
- $S(0)z = z$
- $t_n \to t, z_n \to z \implies S\left(t_n\right) z_n \to S(t)z$
- $S(t)S(s)z = S(t+s)z, \; t, s \geq 0$

Orbit

We define the orbit of a solution starting at z as

$$\gamma(z) = \{S(t)z \,|\, t \geq 0\}$$

Types of Trajectories

Suppose solutions are unique in both directions. Then for each $z \in \Lambda$, the solutions $S(t)z$ is of exactly one of the following 3 types:

1. $S(t)z$ is one-to-one ($S\left(t_1\right) z \neq S\left(t_2\right) z$ if $t_1 \neq t_2$)
2. $S(t)z$ is periodic of minimal period $T > 0$ ($S(t+T)z \equiv S(t)z \,\forall\, t \geq 0$)
3. $S(t)z$ is constant ($S(t)z \equiv z \,\forall\, t \geq 0$)

Proof. Suppose $S(t)z$ is not of type 1. Then $\exists\, t_1 < t_2 \ni S\left(t_1\right) z = S\left(t_2\right) z$. Then we must have

$$(6) \quad S(t)z \equiv S(t+\tau)z \;\forall\, t \geq 0, \;\; \tau = t_2 - t_1$$

This shows that $S(t)z$ is periodic.

$$\therefore S(t)z \text{ is periodic.}$$

Now, define $T \equiv \inf\{\tau > 0 \,|\, (6) \text{ holds true}\} > 0$. We want to show that $S(t+T)z = S(t)z \;\forall\, t \geq 0$. Let $\tau_n \to T$ and $\tau_n \geq T$. Then

$$\begin{aligned} S(t+T)z &= \lim_{n\to\infty} S(t+\tau_n)\,z \\ &= \lim_{n\to\infty} S(t)z \quad \text{(by (6))} \\ &= S(t)z \end{aligned}$$

This shows that

$$S \text{ is periodic with minimal period } T$$

Now, consider when $T = 0$. Then $\exists\, \tau_n \to 0$ which satisfies (6). If we take $t = 0$ in $\lim_{n\to\infty} S(t+\tau_n)\,z$ then we get (for 'large' n),

$$\begin{aligned} S(\tau_n)\,z = z \quad &\Longrightarrow \quad S(\tau_n)\,z - z = 0 \\ &\Longrightarrow \quad \frac{S(\tau_n)\,z - z}{\tau_n} = 0 \end{aligned}$$

and we see that $\lim_{n\to\infty} \frac{S(\tau_n)z - z}{\tau_n} = S'(0)z = f(S(0)z) = f(z) = 0 \implies S(t)z = z$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Omega Limit Set

The omega limit set ω of $S(t)z$ is the set of points $x \in \Lambda \ni \exists\, t_n \to \infty \ni S(t_n)\,z \to x$ as $n \to \infty$ and note that t_n must be an increasing sequence (not strict).

Properties of Omega Limit Set

1. $\omega(z)$ is closed
2. $\omega(z)$ is invariant - $x \in \omega(z) \implies S(t)x \in \omega(z) \;\forall\, t \geq 0$
3. $\omega(z) = \emptyset \iff \lim_{n\to\infty} \|S(t)z\| = \infty$
4. $\omega(z) = \{w\} \iff \lim_{t\to\infty} S(t)z = w$ and furthermore, $f(w) = 0$
5. $\omega(z)$ is bounded $\implies \omega(z)$ is connected

Proof 1. Let $(x_n) \in \omega(z)$ and $x_n \to x$. We WTS $x \in \omega(z)$.
Thus, $\forall\, n \geq 0$, let $t_n > t_{n-1} + 1$ be $\ni \|S(t_n)\,z - x_n\| < \frac{1}{n}$. Then

$$\|S(t_n)\,z - x\| \leq \|S(t_n)\,z - x_n\| + \|x_n - x\| < \frac{1}{n} + \|x_n - x\| \to 0 \text{ as } n \to \infty$$

And thus

$$S(t_n)\,z = x \text{ as } n \to \infty$$

and

$$\therefore x \in \omega(z) \implies \boxed{\omega(z) \text{ is closed.}}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Proof 2. Let $x \in \omega(z)$. We WTS $S(t)x \in \omega(z)\ \forall\, t \geq 0$.

$$x \in \omega(z) \implies x = \lim_{n\to\infty} S(t_n)\, z \quad (t_n \to \infty)$$

So we want to show that $S(t)x \in \omega(z)$ or that $S(t)x = \lim_{n\to\infty} S(s_n)\, z$ for some $s_n \uparrow \infty$.

$$\begin{aligned} S(t)x &= S(t)\left(\lim_{n\to\infty} S(t_n)\, z\right) \\ &= \lim_{n\to\infty} S(t)S(t_n)\, z \\ &= \lim_{n\to\infty} S(t+t_n)\, z \\ &= \lim_{n\to\infty} S(s_n)\, z \quad (\text{take } s_n = t + t_n) \end{aligned}$$

And therefore,

$$S(t)x \in \omega(z)\ \forall\, t \geq 0 \implies \boxed{\omega(z) \text{ is invariant.}}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Proof 3. "$\Longleftarrow$" Trivial since $\omega(z)$ is set of points which solution approaches. If the solution blows up then it is empty.

$$\therefore \omega(z) = \emptyset$$

"$\implies$" Let $\omega(z) = \emptyset$. For contradiction, assume $\lim_{t\to\infty} ||S(t)z|| \neq 0$. Then

$$\exists\, t_n > n \ni \|S(t_n)\, z\| \leq M < \infty \text{ for some } M \in \mathbb{R}$$

But then we have the $S(t_n)\, z$ is bounded which means that $S(t_{n_k})\, z \to x$ since we can find some subsequence of t_n, t_{n_k}, such that this is a convergent sequence (bounded and closed $\implies$ compact). Thus $S(t_{n_k})\, z$ must also converge to something which we call x. But then $x \in \omega(z) = \emptyset$ gives us our contradiction.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Invariant Sets

Let Ω denote the interior of an invariant set and $\bar{\Omega}$ be the actual set.

1. If $\Omega = (0,\infty)^N$, then $\bar{\Omega}$ is invariant $\iff x \geq 0$ and $x_k = 0 \implies f_k(x) \geq 0$
2. If $\Omega = \prod_{i=1}^{N} (a_i, b_i) = \left(\vec{a}, \vec{b}\right)$, then $\bar{\Omega}$ is invariant $\iff \vec{a} \leq x \leq \vec{b}$ and if $\begin{array}{ccc} x_k = a_k & \implies & f_k(x) \geq 0 \\ x_k = b_k & \implies & f_k(x) \leq 0 \end{array}$.
3. If $\Omega = \{x \,|x \cdot \vec{a_i} < c_i,\ \forall\, i = 1, 2, \ldots, m\}$, then $\bar{\Omega}$ is invariant $\iff x \cdot \vec{a}_k = c_k \implies \vec{a}_k \cdot f_k(x) \leq 0$.
4. If $\Omega = \{x \in \Lambda \,|\varphi_i(x) < c_i,\ \forall\, i = 1, 2, \ldots, m\}$, then $\bar{\Omega}$ is invariant $\iff D^{\pm}\varphi_i(x)f(x) \leq 0$ if $\varphi_i(x) = c_i$.

Nullclines

The u_i nullcline, represented as N_{u_i} or N_i is $\{x \,| f_i(x) = 0\}$.

Monotone Flows

Monotone $S(t)z$ is monotone $\iff x \geq y \implies S(t)x \geq S(t)y$

Quasi-positive f is quasi-positive $\iff x \geq 0$ and $x_k = 0 \implies f_k(x) \geq 0$

Also can say this if f is linear and it's off diagonal terms are positive

Quasi-monotone f is quasi-monotone $\iff x \geq y$ and $x_k = y_k \implies f_k(x) \geq f_k(y)$

We can also say this is true if f is $\mathcal{C}^1$ and its Jacobian matrix is q-p.

Theorem In $(\star)$, $S(t)$ is monotone $\iff$ f is qm

Theorem Suppose S is monotone. Then

(1) $z \in \Lambda$ and $f(z) \geq 0 \iff S(t)z \uparrow$ in t

(2) $z \in \Lambda$ and $f(z) \leq 0 \iff S(t)z \downarrow$ in t

Analyzing Systems of ODEs

1. Check invariance of system (f is qp)
2. Find critical points
3. Check to see if $S(t)$ is monotone (check if f is qm)
4. Find $\hat{z} \ni f(\hat{z}) < 0$
5. Determine the stability of the other critical points (try to find $\hat{z}$ such that $f(\hat{z})$ is positive or negative) or look at $Df(0,0)$ if for example the origin is the critical point in mind. Find the ev of this matrix.

Lyapunov Function for $(\star)$

A Lyapunov function for $(\star)$ must satisfy the following

1. V is p-d. I.e. $V[0] = 0$ and $V[x] > 0$ if $x \neq 0$
2. V is locally lipschitz (lipschitz on bounded sets). I.e. $|V[x] - V[y]| \leq L_R ||x - y||$ if $||x||, ||y|| \leq R$
3. $D^{\pm}V_{(\star)}[x] \leq 0 \; \forall \, x \in \Lambda$
4. $\forall \, r > 0 \, \exists \, a_R \uparrow$ strictly and continuous and $L_R > 0 \ni a(||x||) \leq V[x] \leq L_R ||x|| \, \forall \, ||x|| \leq R$

Lyapunov's (S) Theorem - Let $f(0) = 0$. If V is a Lyapunov function for $(\star)$ as defined above, then the CP 0 is **(S)**.

Proof. Let $0 < \epsilon < R$, $t_0 \geq 0$ be given. Then we have $a(||x||) \leq V[x] \leq b \cdot ||x|| \; \forall \; x \in \Lambda$, $||x|| < R$

$\mathbf{D}V_{(\star)}[t, u(t)] \leq 0 \implies V$ is non-increasing. So then we get:

$$V[u(t)] \leq V[u(t_0)]$$

Choose $\delta = \delta(\epsilon) > 0 \ni b \cdot \delta < a(R)$ (i.e., $b \cdot \delta \in \text{Rng}(a)$) and $a^{-1}(b \cdot \delta) < \epsilon$. (Remember $||u(t_0)|| < \delta$)

So $a\left(||u(t)||\right) \leq V\left[u(t)\right] \leq V\left[u\left(t_0\right)\right] \leq b \cdot ||u\left(t_0\right)|| < b \cdot \delta \implies a\left(||u(t)||\right) < b \cdot \delta \implies ||u(t)|| < a^{-1}\left(b \cdot \delta\right) < \epsilon$

$$\therefore \text{The zero solution to } (\star) \text{ is } \textbf{(S)} .$$

QED

Lyapunov's (AS) Theorem - Suppose $\eta > 0$ and $V : ||x|| \leq \eta \to [0, \infty)$ is p.d. and locally lipschitz. If $DV_{(\star)}[x] \leq -W[x] \, \forall \, x \leq \eta$ where $W : ||x|| \leq \eta \to [0, \infty)$ is p.d., then 0 is **(AS)** CP for $(\star)$.

Proof. $(\star)$ is **(S)** by the previous theorem so we want to show **(AS)**.

Assume $a(||x||) \leq V[x] \leq b \cdot ||x||$ and $\bar{a}(||x||) \leq W[x] \leq \bar{b} \cdot ||x|| \; \forall \, x \in \Lambda, \, ||x|| \leq R$.

Let η be $\ni$ if $||u\left(t_0\right)|| < \eta$ then $\bar{a}(||u(t)||) \leq \bar{a}(R) \; \forall \, t \geq t_0 \geq 0$. We claim the following:

$$||u\left(t_0\right)|| < \eta \implies \lim_{t \to \infty} V[u(t)] = 0$$

For contradiction, suppose $||u\left(t_0\right)|| < \eta$ but $\lim_{t \to \infty} V[u(t)] \neq 0$. Since $V[u(t)]$ is positive-definite, this is equivalent to saying $\lim_{t \to \infty} V[u(t)] > 0$.

$$\mathbf{D}V_{(\star)}[u(t)] \leq -W[u(t)] < 0 \implies V[u(t)] \text{ is decreasing}$$

Thus we have $V[u(t)]$ is decreasing and $\lim_{t \to \infty} V[u(t)] > 0$ which means $\exists \, \alpha > 0 \ni V[u(t)] \geq \alpha$. Then we have $b \cdot ||u(t)|| \geq V[u(t)] \geq \alpha \implies ||u(t)|| \geq \frac{\alpha}{b}$.

$$\begin{aligned} W[u(t)] \geq \bar{a}(||u(t)||) \geq \bar{a}\left(\frac{\alpha}{b}\right) &\implies \mathbf{D}V[u(t)] \leq -W[u(t)] \leq -\bar{a}\left(\frac{\alpha}{b}\right) \\ &\implies V[u(t)] - V\left[u\left(t_0\right)\right] \leq -\bar{a}\left(\frac{\alpha}{b}\right) \cdot \left(t - t_0\right) \\ &\implies V[u(t)] \leq V\left[u\left(t_0\right)\right] - \bar{a}\left(\frac{\alpha}{b}\right) \cdot \left(t - t_0\right) \to -\infty \text{ as } t \to \infty \\ &\implies V[u(t)] \to -\infty \text{ which is a contradiction} \end{aligned}$$

Thus, $\lim_{t \to \infty} V[u(t)] = 0$.

Now,

$$\begin{aligned} &a(||u(t)||) \leq V[u(t)] \to 0 \text{ as } t \to \infty \\ &\implies ||u(t)|| \leq a^{-1}\left(V\left[u\left(t\right)\right]\right) \to 0 \text{ since } V[u(t)] \to 0 \end{aligned}$$

since $V[u(t)] \to 0$ and $a(0) = 0 \implies a^{-1}(0) = 0$ all as $t \to \infty$.

$$\therefore \lim_{t \to \infty} ||u(t)|| = 0 \implies \text{The zero solution to } \boldsymbol{(\star)} \text{ is } \textbf{(AS)}.$$

QED

Basin of Attraction

Let $\hat{B}(w)$ of a CP w be the set of all $z \in \Lambda \ni S(t)z \to w$.

Theorem Suppose $0 \in \Lambda$ and $\exists$ open $U \subset \mathbb{R}^N \ni 0 \in U$. $V : U \to [0, \infty)$ is p.d. and locally lipschitz and in addition $V[x] \to \infty$ as $||x|| \to \infty$, $x \in U$. Then if $\exists$ p.d. $W : U \to [0, \infty) \ni$

$$DV_{(\star)}[x] \leq -W[x] \, \forall \, x \in U \cap \Lambda$$

and $x \in U \cap \Lambda$ and $S(t)z \in U \cap \Lambda$, then $S(t)z \to 0$ as $t \to \infty$ $\Big(\text{i.e. } z \in \hat{B}(0)\Big)$.

Stability by Linearization

Check Jacobian at CPs and find values. If $\text{Re}(\lambda) < 0 \, \forall \, \lambda \in \sigma(A)$, then **(S)** about the CP where $A = \mathbf{D}f(0)$ and about the CP $f(x) = Ax$.

Theorem These are equivalent:

1. A is **(AS)**
2. $\text{Re}(\lambda) \leq -\rho < 0 \, \forall \, \lambda \in \sigma(A)$ and if $\text{Re}(\lambda) = -\rho$ then λ is simple (algebraic multiplicity is equal to geometric multiplicity)
3. $\left\| e^{tA} \right\| \leq k e^{-\rho t} \, \forall \, t \geq 0$ $(k \geq 1)$
4. $\exists$ equivalent norm $||| \cdot |||$ $(||x|| \leq |||x||| \leq k||x||)$ such that $\left|\left|\left| e^{tA} \right|\right|\right| \leq e^{-\rho t}$ where $|||x||| \equiv \sup_{t \geq 0} \left\{ e^{\rho t} \left\| e^{tA} x \right\| \right\}$.
5. $\hat{M}_+[x, Ax] \leq -\rho ||x|| \, \forall \, x \in \mathbb{R}^N$

Define

$$(\star)' \quad u' = Au + g(u)$$

where $f(0) = 0$ and $A = \mathbf{D}f(0)$ and we have $||g(x)|| \leq e||x||$ if $||x|| < \delta(\epsilon)$.

Theorem If A is **(AS)**, then so is $(\star)'$.

Basic Invariance Principle / Lasalle's Invariance Principle

Suppose U is an open subset of $\mathbb{R}^N$, $0 \in U$, $V : U \cap \Lambda \to [0, \infty)$ is p.d. and locally lipschitz and $V[x] \to \infty$ as $||x|| \to \infty$. Let $\Gamma \subset U \cap \Lambda \ni 0 \in \Gamma$ and

$$DV_{(\star)}[x] \leq 0 \; \forall \, x \in \Gamma$$

Set $M = \left\{ x \,\middle|\, DV_{(\star)}[x] = 0 \right\}$. If $z \in \Gamma$ and $S(t)z$ remains in $\Gamma \, \forall \, t \geq 0$, then

$$\omega(z) \subset M$$

In particular, since $\omega(z)$ is invariant, $\omega(z) \subset N$, where N is the largest invariant subset of M.