MA 515
Test 2 Study Guide

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November 8, 2010

Earlier Material

Example Banach spaces.

- $\mathbb{R}^n$ and $\mathbb{C}^n$ with norm $\|x\|_2$
- l^p with norm $\|x\|_p$
- l^∞ with norm $\|x\| = \sup_{i \in \mathbb{N}} |x_i|$
- $\mathcal{C}[a, b]$ with norm $\|x\| = \max_{t \in [a,b]} |x(t)|$

Example incomplete normed vector spaces.

- $\mathbb{Q}$ with norm $|x|$
- $\mathbb{P}[a, b]$ (polynomials) with norm $\|x\| = \max_{t \in [a,b]} |x(t)|$
- $\mathcal{C}[a, b]$ with norm $\|x\| = \int_a^b |x(t)| dt$
  - Completion of this is given by $L^2[a, b] = \left\{ f : [a, b] \to \mathbb{R} \mid \int_a^b |f(t)|^2 dt < \infty \right\}$

Metrics obtained from norms. Must satisfy that $d(x + a, y + a) = d(x, y)$ for all $x, y, a \in X$ and $d(\alpha x, \alpha y) = |\alpha| \cdot d(x, y)$ for all $x, y \in X$ and $\alpha \in \mathbb{K}$.

Section 2.3. Further Properties of Normed Spaces

Theorem (subspace completeness). A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Convergent sequence. $x_n \to x$ in X if and oly if $\|x_n - x\| \to 0$ as $n \uparrow \infty$.

Convergent series. Let (x_n) be a sequence and $s_n = x_1 + \cdots + x_n$. If $\|s_n - s\| \to 0$ for some s, then $\sum_{i=1}^{\infty} x_i$ converges to s.

Absoulte convergence. Series obtained from (x_n) absolutely converges if and only if $\|x_1\| + \|x_2\| + \cdots$ converges.

Remark. Absolute convergence $\implies$ convergence if and only if X is Banach.

Completion of Arbitrary Normed Space

Section 2.4. Finite Dimensional Normed Spaces

Lemma. If $\{e_i\}_{i=1,\dots,n}$ is a linearly independent set in X, then $\exists\ M, c$ such that

$$c\sum_{i=1}^{n}|\alpha_i| \le \left\|\sum_{i=1}^{n}\alpha_i e_i\right\| \le M\sum_{i=1}^{n}|\alpha_i|$$

Proof. Note that if $\sum_{i=1}^{n}|\alpha_i| = 0$ then this is vacuously satisfied. Assume that $\sum_{i=1}^{n}|\alpha_i| \neq 0$.
First see that for $M = \max_{i=1,\dots,n}\|e_i\|$ and the triangle inequality that

$$\left\|\sum_{i=1}^{n}\alpha_i e_i\right\| \le \sum_{i=1}^{n}\|\alpha_i e_i\| = \sum_{i=1}^{n}|\alpha_i|\,\|e_i\| \le M\sum_{i=1}^{n}|\alpha_i|$$

Note that if

$$c\sum_{i=1}^{n}|\alpha_i| \le \left\|\sum_{i=1}^{n}\alpha_i e_i\right\| \implies c \le \left\|\sum_{i=1}^{n}\frac{\alpha_i}{\sum_{i=1}^{n}|\alpha_i|}e_i\right\|$$

and thus defining $\beta_i = \frac{\alpha_i}{\sum_{i=1}^{n}|\alpha_i|}$ we know that $\sum_{i=1}^{n}|\beta_i| = 1$. Thus we equivalently may show that $\|\sum_{i=1}^{n}\beta_i e_i\| \ge c > 0$ for any $\{\beta_i\}_{i=1,\dots,n}$ satisfying $\sum_{i=1}^{n}|\beta_i| = 1$. Let $M = \{x = (x_1,\dots,x_n) \in \mathbb{K}^n \mid \sum_{i=1}^{n}|x_i| = 1\}$.
For contradiction assume this is not true. That is, there is a sequence $\left\{\beta^{(m)}\right\}_{m\in\mathbb{N}}$ where $\beta^{(m)} = \left\{\beta_i^{(m)}\right\}_{i=1,\dots,n}$ satisfying $\left\|\sum_{i=1}^{n}\beta_i^{(m)}e_i\right\| \to 0$ as $m \uparrow \infty$ with $\sum_{i=1}^{n}\left|\beta_i^{(m)}\right| = 1$ for all $m \in \mathbb{N}$. Note that this last condition implies that $\left|\beta_i^{(m)}\right| \le 1$ for all $i = 1,\dots,n$ and $m \in \mathbb{N}$. Then by the Bolzano-Wieirstrass Theorem we have that $\beta^{(m)}$ has a convergent subsequence $\beta^{(m_k)} \to \gamma \in M$ (it is in M because M is closed). Thus $\sum_{i=1}^{n}|\gamma_i| = 1$. But note that

$$\sum_{i=1}^{n}\beta_i^{(m)}e_i \to \sum_{i=1}^{n}\gamma e_i \text{ as } m\uparrow\infty \qquad \text{and} \qquad \sum_{i=1}^{n}\beta_i^{(m)}e_i \to 0 \text{ as } m\uparrow\infty$$

and this directly implies that $\sum_{i=1}^{n}\gamma e_i = 0 \implies \gamma_i = 0$ for all $i = 1,\dots,n$ by the linear independence of $\{e_i\}_{i=1,\dots,n}$. This contradicts that fact that $\sum_{i=1}^{n}|\gamma_i| = 1$.

$$\therefore \exists\ c > 0 \text{ such that } c\sum_{i=1}^{n}|\alpha_i| \le \left\|\sum_{i=1}^{n}\alpha_i e_i\right\|$$

$\mathbb{Q.E.D.}$

Completeness of Finite Dimensional Normed Vector Spaces

Let X be a finite dimensional subspace of V, a normed vector space. Then X is complete.

Proof. Let $\dim X = d \implies X$ has a Hamel basis $\{e_i\}_{i=1,\dots,d}$. Notice a priori we have $\exists\ c > 0$ such that

$$c\sum_{i=1}^{d}|\alpha_i| \le \left\|\sum_{i=1}^{d}\alpha_i e_i\right\| \qquad \forall\ \alpha = (\alpha_i) \in \mathbb{K}^d$$

Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be Cauchy and thus

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \|x_n - x_m\| < \epsilon \cdot c \text{ if } n, m \geq N$$

We can write each x_n as

$$x_n = \sum_{i=1}^{d} x_i^{(n)} e_i \text{ for some } x_i^{(n)} \in \mathbb{K}$$

Thus

$$\begin{aligned} \|x_n - x_m\| < \epsilon \cdot c \quad &\implies \quad c \cdot \sum_{i=1}^{d} \left| x_i^{(n)} - x_i^{(m)} \right| \leq \left\| \sum_{i=1}^{d} \left(x_i^{(n)} - x_i^{(m)} \right) e_i \right\| < \epsilon \cdot c \\ &\implies \quad \sum_{i=1}^{d} \left| x_i^{(n)} - x_i^{(m)} \right| < \epsilon \end{aligned}$$

and therefore for each $i = 1, \ldots, d$ we have $\left| x_i^{(n)} - x_i^{(m)} \right| < \epsilon$ and therefore each $\left\{ x_i^{(n)} \right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ for all $i = 1, \ldots, d$ and since $\mathbb{K}$ is complete, we thus have that each $x_i^{(n)} \to \eta_i \in \mathbb{K}$ for each $i = 1, \ldots, d$.

We claim that $x_n \to x = \sum_{i=1}^{d} \eta_i e_i$. See that

$$\|x_i - x\| = \left\| \sum_{i=1}^{d} \left(x_i^{(n)} - \eta_i \right) e_i \right\| \leq M \cdot \sum_{i=1}^{d} \left| x_i^{(n)} - \eta_i \right| \to 0 \quad \text{ as } n \uparrow \infty$$

and this completes the proof.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Theorem. If X is a finite dimensional normed vector space with norms $\|\cdot\|_1, \|\cdot\|_2$ and basis $\{e_i\}_{i=1,\ldots,d}$, then $\exists\, a, b \geq 0$ such that $a \cdot \|x\|_2 \leq \|x\|_1 \leq b \cdot \|x\|_2$.

Proof. Note that for all $x \in X$ and for $k = 1, 2$ we have $\exists\, M, c > 0$ such that

$$c_k \cdot \sum_{i=1}^{d} |x_i| \leq \underbrace{\left\| \sum_{i=1}^{d} x_i e_i \right\|_k}_{\|x\|_k} \leq M_k \cdot \sum_{i=1}^{d} |x_i|$$

Then

$$\frac{c_1}{M_2}\|x\|_2 \leq \frac{M_2}{M_2} c_1 \sum_{i=1}^{d} |x_i| \leq \|x\|_1 \leq M_1 \sum_{i=1}^{d} |x_i| \cdot \frac{c_2}{c_2} \leq \frac{M_1}{c_2}\|x\|_2$$

Section 2.5. Compactness and Finite Dimension

Compactness. If $Y \subseteq (X, d)$ a metric space, then K compact $\iff$ all sequences in K have a convergent subsequence (in K).

Theorem. In a finite dimsional normed space X, any $M \subseteq X$ is compact if and only if M is closed and bounded.

Proof. "$\implies$" Let $x \in \bar{M} \implies \exists\, x_n \to x$. M is compact so it has a convergent subsequence, converging in M, and thus $x \in M \implies \bar{M} \subseteq M$ and a prior we knew $M \subseteq \bar{M}$ and thus $M = \bar{M}$ and M is closed. For contradiction assume M is not bounded. Then $\exists\, (y_n)$ such that for any fixed $b \in M$ we have $d(y_n, b) > n$ for all $n \in \mathbb{N}$. But then this could not have a convergent subsequence.

"$\Longleftarrow$" Let $M \subseteq X$ be closed and bounded. Suppose $\dim X = n$ and $\{e_i\}_{i=1,\dots,n}$ is a basis for X. Let (x_m) be a sequence in M and thus for fixed $m \in \mathbb{N}$ we have that

$$x_m = x_1^{(m)} e_1 + \cdots + x_n^{(m)} e_n$$

and since M is bounded then so is (x_m) and thus $\|x_m\| \leq k$ for some $k \in \mathbb{K}$ for all $m \in \mathbb{N}$. Then by a previous observation,

$$k \geq \|x_m\| = \left\| \sum_{i=1}^{n} x_i^{(m)} e_i \right\| \geq c \cdot \sum_{i=1}^{n} \left| x_i^{(m)} \right| \quad \forall\, m \in \mathbb{N}$$

Then for fixed i, $\left\{ x_i^{(m)} \right\}_{m \in \mathbb{N}}$ is bounded in $\mathbb{K}$ and thus each $x_i^{(m_k)} \to \eta_i$ as $m_k \uparrow \infty$ for fixed $i = 1, \dots, n$ by the Bolzano-Wieirstrass Theorem. I claim that $x_{m_k} \to z = \sum_{i=1}^{n} \eta_i e_i$. See that

$$\|x_{m_k} - z\| = \left\| \sum_{i=1}^{n} x_i^{(m_k)} e_i - \sum_{i=1}^{n} \eta_i e_i \right\| = \left\| \sum_{i=1}^{n} \left(x_i^{(m_k)} - \eta_i \right) e_i \right\| \leq M \sum_{i=1}^{n} \left| x_i^{(m_k)} - \eta_i \right| \to 0$$

completing the proof. Further since M is closed and $\{x_{m_k}\} \subseteq M$ then $z \in M$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Riesz Lemma

Let $Y \subsetneq X$ (normed vector space) be a closed subspace. Then $\forall\, \theta \in (0,1)$, $\exists\, z \in S(0,1) \subseteq X$ (unit vector) such that $d(z, Y) > \theta$.

Proof. Let $x_0 \in X - Y$. Then

$$d = \inf_{y \in Y} d(z, y) = d(x_0, Y) > 0$$

Note that this must be strictly positive as otherwise we would have $\inf_{y \in Y} d(z, y) = 0 \implies \exists\, \{y_n\} \subseteq Y$ such that $d(x_0, y_n) \to 0$ and then $y_n \to x_0$ but $x_0 \notin Y$ contradicts closedness.

Trivially see that for all $0 < \theta < 1$ that $\frac{1}{\theta} > 1$ and thus $d < \frac{1}{\theta} d \implies \inf_{y \in Y} d(x_0, y) < \frac{1}{\theta} d \implies \exists y_0 \in Y$ such that $d < \underbrace{d(x_0, y_0)}_{\|x_0 - y_0\|} < \frac{1}{\theta} d \implies \theta \|x_0 - y_0\| < d$.

Take $z = \frac{x_0 - y_0}{\|x_0 - y_0\|}$ and let $y \in Y$. Then

$$\begin{aligned}
\|z - y\| &= \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \right\| = \frac{1}{\|x_0 - y_0\|} \|x_0 - \underbrace{(y_0 + y\|x_0 + y_0\|)}_{\in Y}\| \\
&= \frac{1}{\|x_0 - y_0\|} \|x_0 - y'\| \quad \text{for some } y' \in Y \\
&\geq \frac{1}{\|x_0 - y_0\|} d
\end{aligned}$$

Thus

$$d(z, Y) = \inf_{y \in Y} d(z, y) \geq \frac{d}{\|x_0 - y_0\|} > \frac{\theta \|x_0 - y_0\|}{\|x_0 - y_0\|} = \theta$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Applications of Riesz Lemma

First one. X is a normed vector space. $\tilde{B}(0,1)$ is compact $\implies \dim X < \infty$.

Proof. Assume X is a normed vector space with $\tilde{B}(0,1)$ compact. Assume for contradiction that $\dim X = \infty$. Let $x_1 \in X, x_1 \neq 0$. Then

$$\begin{aligned}
M_1 &= \text{span}\{x_1\} \subsetneq X \text{ is a finite dimensional subspace and thus closed} \\
&\quad \implies \exists\, x_2 \in S(0,1) \subsetneq \tilde{B}(0,1) \ni d(x_2, M_1) > \frac{1}{2} \text{ by R. Lemma} \\
M_2 &= \text{span}\{x_1, x_2\} \subsetneq X \implies \exists\, x_3 \in S(0,1) \ni d(x_3, M_2) > \frac{1}{2} \\
&\vdots \\
M_n &= \text{span}\{x_1, \ldots, x_n\} \subsetneq X \implies \cdots \\
&\vdots
\end{aligned}$$

Now consider $\{x_n\} \subseteq S(0,1) \subsetneq \tilde{B}(0,1)$ compact $\implies \exists\, \{x_{n_k}\} \subseteq S(0,1)$ such that $x_{n_k} \to y \in \tilde{B}(0,1)$. Note that then $\{x_{n_k}\}$ is Cauchy because it converges and thus

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \|x_n - x_m\| < \epsilon \text{ if } n, m \geq N$$

WLOG let $m > n$. Then $x_n \in M_{m-1} = \text{span}\{x_1, \ldots, x_{m-1}\} \implies \|x_n - x_m\| \geq d(x_m, M_{m-1}) > \frac{1}{2}$. But then we have that for all $\epsilon > 0$,

$$\frac{1}{2} < \|x_n - x_m\| < \epsilon$$

giving our contradiction.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Second one. $Y \subsetneq X$ subspace and $\exists\ 0 < r < 1$ such that $d(x, Y) < r$ for all $x \in S(0,1) \implies Y$ dense in X (i.e. $\bar{Y} = X$).

Proof. Suppose for contradiction that Y is not dense in X. That is, $\bar{Y} \subsetneq X$. Using the Riesz lemma with $r = \theta \implies \exists\, x_0 \in S(0,1)$ such that $d(x_0, \bar{Y}) > r$. But $r < d(x_0, \bar{Y}) \leq d(x_0, y) < r$ for all $y \in Y \implies r < r$ giving our contradiction.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Section 2.6. Linear Operators

Linear operator. A linear operator T is an operator $T : X \to Y$ with respective norms $\|\cdot\|_X, \|\cdot\|_Y$ where $x \mapsto Tx$ and assume X and Y have scalar fields. T satisfies $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$. We also define $\mathcal{D}(T)$ to be the domain of T, $\mathcal{R}(T)$ to be the range of T, and $\mathcal{N}(T)$ denotes the null space of T given by $\mathcal{N}(T) = \ker T = \{x \in X \mid Tx = 0\}$.

Examples.

- Identity operator. $I_X : X \to X$ defined by $I_X x = x$.
- Zero operator. $0 : X \to Y$ defined by $0x = x$.
- Differentiation operator. Let X be the set of polynomials on $[a,b]$. Define a linear operator T by $Tx(t) = x'(t)$, $T : X \to X$.

- Integration operators. A linear operator $T : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined by $Tx(t) = \int_a^t x(\tau)d\tau$.
- Multiplication by t. A linear operator $T : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined by $Tx(t) = tx(t)$.

Theorem (range and null space). Let T be a linear operator. Then

- The range $\mathcal{R}(T)$ is a vector space.
- If $\dim \mathcal{D}(T) = n < \infty$, then $\mathcal{D}(T) \leq n$.
- The null space $\mathcal{N}(T)$ is a vector space.

Theorem (inverse operator). Let X, Y be vector spaces, both with the same scalar field $\mathbb{K}$. Let $T : X \to Y$ be a linear operator with domain $\mathcal{D}(T) \subseteq X$ and range $\mathcal{R}(T) \subseteq Y$. Then

- The inverse $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ exists if and only if

$$Tx = 0 \qquad \Longrightarrow \qquad x = 0$$

- If T^{-1} exists, it is a linear operator.
- If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}(T) = \dim \mathcal{D}(T)$.

Section 2.7. Bounded and Continuous Linear Operators

Norm of linear operator. $T : X \to Y$ has norm given by $\|T\|_{op.} = \sup_{x \in X, x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} < \infty$.

Bounded linear operator. A bounded linear operator has $\|T\|_{op.} < \infty$ and further note this directly implies that $\|Tx\|_Y \leq c \cdot \|x\|_X$ for some $c \geq 0$ (namely $c = \|T\|_{op.}$).

Lemma. We may equivalently write $\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\|$.

Examples.

- Identity operator I is bounded and have $\|I\| = 1$.
- Zero operator 0 is bounded and has $\|0\| = 0$.
- Differential operator T is unbounded (consider polynomials $x_n(t) = t^n$.
- Integral operator T is linear and bounded when $Tx(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$ and $|k(t,\tau)| \leq k_0$ for all $(t,\tau) \in [0,1] \times [0,1]$ and $\|T\|_{op.} = k_0$.
- Matrix operator $T : \mathbb{R}^n \to \mathbb{R}^r$ and efined for some $r \times n$ matrix A by $Tx = Ax$ is bounded and have $\|T\|_{op.} = \sqrt{\sum_{i=1}^r \sum_{j=1}^n a_{ij}^2}$.

Theorem (finite dimension). If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof. Suppose $\dim X = n$ and thus X has Hamel basis given by $\{e_i\}_{i=1,\ldots,n}$ and thus for any $x \in X$ we can write $x = \sum_{i=1}^n x_i e_i$. Then

$$\|Tx\| = \left\| \sum_{i=1}^n x_i T e_i \right\| \leq \sum_{i=1}^n |x_i| \|Te_i\| \leq \max_{i=1,\ldots,n} \|Te_i\| \sum_{i=1}^n |x_i|$$

We know that

$$\sum_{i=1}^{n} |x_i| \leq \frac{1}{c} \left\| \sum_{i=1}^{n} x_i e_i \right\| = \frac{1}{c} \|x\|$$

and therefore

$$\|Tx\| \leq \left(\frac{1}{c} \max_{i=1,\ldots,n} \|Te_i\| \right) \|x\|$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Theorem (continuity and boundedness). Let $T : X \to Y$ be a linear operator where X, Y are normed spaces. Then

- T continuous if and only if T is bounded.
- If T is continuous at a single point then it is continuous everywhere.

Proof. "$\Longleftarrow$" Assume T is bounded. Let $\epsilon > 0$ and $\|x - x_0\| < \frac{\epsilon}{\|T\|}$ and thus $\|Tx - Tx_0\| \leq \|T\| \|x - x_0\| < \epsilon$.

"$\implies$" Assume T continuous at $x_0 \in X$ and thus $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that $\|Tx - Tx_0\| \leq \epsilon$ for all $x \in X$ with $\|x - x_0\| \leq \delta$. Take any $y \in X$ and let

$$x = x_0 + \frac{\delta}{\|y\|} y \implies x - x_0 = \frac{\delta}{\|y\|} y \implies \|x - x_0\| = \delta$$

Then

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T\left(\frac{\delta}{\|y\|} y \right) \right\| = \frac{\delta}{\|y\|} \|Ty\|$$

and thus since $\frac{\delta}{\|y\|} \|Ty\| = \|Tx - Tx_0\| \leq \epsilon \implies \|Ty\| \leq \frac{\epsilon}{\delta} \|y\|$ and thus T is bounded with $\|T\|_{op.} = \frac{\epsilon}{\delta}$.

Continuity of T at a point implies boundedness of T by the second part above, implying continuity.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Theorem (bounded linear extension). Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator, where $\mathcal{D}(T) \subseteq X$ (normed space) and Y is Banach space. Then T has an extension

$$\bar{T} : \overline{\mathcal{D}(T)} \to Y$$

where $\bar{T}$ is a bounded linear operator of norm $\|\bar{T}\| = \|T\|$.

Proof. Consider $x \in \overline{\mathcal{D}(T)} \implies \exists\ \{x_n\} \subseteq \mathcal{D}(T)$ such that $x_n \to x$. T is linear and bounded so

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\| \|x_n - x_m\| \to 0 \qquad \implies \qquad \{Tx_n\}_{n \in \mathbb{N}} \subseteq \mathcal{R}(T) \text{ is Cauchy}$$

Since Y complete then $Tx_n \to y \in Y$. Thus we have a definition for $x \in \overline{\mathcal{D}(T)}$, $\bar{T}x = y$. Is this well-defined? Let $\{x_n\}, \{y_n\} \subseteq \mathcal{D}\ (T)$ such that $x_n \to x$ and $y_n \to x$ and thus we WTS $Tx = Ty \implies \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Ty_n$. But $Tx_n - Ty_n = T(x_n - y_n) \to T0 = 0 \implies Tx_n = Ty_n$ for all $n \in \mathbb{N}$.

Next we WTS that 1) $\bar{T}$ linear, 2) $\bar{T}$ bounded, 3) $\bar{T}\ |_{\mathcal{D}(T)} = T$, 4) $\|\bar{T}\| = \|T\|$. all are trivial.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Section 2.8. Linear Functionals

Section 2.9 Linear Operators and Functionals on Finite Dimensional Spaces

Unique representation of linear operators. Let $T : X \to Y$ where X, Y are normed vector spaces with respective bases $\{e_i\}_{i=1,\dots,n} \subseteq X$ and $\{b_i\}_{i=1,\dots,r} \subseteq Y$. For any $x \in X$ we have $x = \sum_{i=1}^n x_i e_i$ and it has the image $y = Tx = \sum_{i=1}^n x_i T e_i$ and thus see that $y_k = Te_k$ for $i = 1, \dots, r$. Further, we may write each $y \in Y$ as $y = \sum_{i=1}^r y_i b_j$ and thus $y = \sum_{i=1}^n x_i T e_i = \sum_{i=1}^n x_i \sum_{j=1}^r \tau_{ji} b_j = \sum_{j=1}^r \left(\sum_{i=1}^n \tau_{ji} x_i\right) b_j$.

Section 2.10. Normed Spaces of Operators. Dual Space

Space of bounded linear operators. X is a normed vector space, and Y is a Banach space. Then $B(X,Y) = \{T : X \to Y \mid T \text{ bounded and linear}\}$ is Banach.

Proof. Let $\{T_n\}_{n\in\mathbb{N}} \subseteq B(X,Y)$ be Cauchy. WTS $T_n \xrightarrow{\|\cdot\|_{op.}} T \in B(X,Y)$. We have

$$\forall\ \epsilon > 0,\ \exists\ N_\epsilon \in \mathbb{N} \text{ such that } \|T_n - T_m\|_{op.} < \frac{\epsilon}{2} \text{ if } n, m \geq N_\epsilon$$

and then for any fixed $x \in X$ we can also have

$$\forall\ \epsilon > 0,\ \exists\ N_{x,\epsilon} \in \mathbb{N} \text{ such that } \|T_n - T_m\|_{op.} < \frac{\epsilon}{2\|x\|} \text{ if } n, m \geq N_{x,\epsilon}$$

and then we have that $\{T_n x\}_{n\in\mathbb{N}} \subseteq Y$ is Cauchy since

$$\|T_n x - T_m x\| \leq \|T_n - T_m\|_{op.}\|x\| < \frac{\epsilon}{2\|x\|}\|x\| = \frac{\epsilon}{2} < \epsilon \text{ if } n, m \geq N_{x,\epsilon}$$

Then we have that $T_n x \to Tx \in Y$ as $n \uparrow \infty$. Thus we have a natural definition for $Tx = \lim_{n\to\infty} T_n x$ for all $x \in X$. We want to show $T \in B(X,Y)$. I.e. we want to show that T is linear and bounded. T is linear is trivial. We'll show the boundedness of T.

See that $\{T_n\}$ Cauchy $\implies$ $\{\|T_n\|\} \subseteq \mathbb{R}$ is Cauchy and thus $\|T_n\| \to \alpha \in \mathbb{R}$ since $\mathbb{R}$ is complete. See that

$$\|T_n x\| \leq \|T_n\|\|x\| \implies \lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} \|T_n\|\|x\| \underbrace{\implies}_{cont.\ of\ \|\cdot\|} \|Tx\| \leq \alpha\|x\|$$

showing T is bounded.

Last we must show that $T_n \to T$, that is $\|T_n - T\| \to 0$ as $n \uparrow \infty$. Note that since $\|T_n - T\| = \sup_{x\in X, \|x\|=1} \|T_n x - Tx\|$ and thus it suffices to show that for all $\|x\| = 1$ we have $\|T_n x - Tx\| \to 0$. See that

$$\begin{aligned}
\|T_n x - Tx\| &= \left\|T_n x - \lim_{m\to\infty} T_m x\right\| \underbrace{=}_{cont.\ of\ \|\cdot\|} \lim_{m\to\infty} \|T_n x - T_m x\| \leq \lim_{m\to\infty} \|T_n - T_m\|_{op.} \cdot \underbrace{\|x\|}_{=1} \\
&= \lim_{m\to\infty} \|T_n - T_m\|_{op.} < \lim_{m\to\infty} \frac{\epsilon}{2} \quad \text{if } n, m \geq N_\epsilon \\
&< \epsilon
\end{aligned}$$

and $m \geq N_\epsilon$ trivially since $m \uparrow \infty$. Thus we have shown that

$$\forall\ \epsilon > 0,\ \|T_n x - Tx\| < \epsilon \text{ if } n \geq N_\epsilon$$

and thus $T_n \to T$ follows.

Dual Spaces (up to isomorphism)

1. $(l^p)' \cong l^q$ for $\frac{1}{p}+\frac{1}{q}=1$ with $1<p,q<\infty$

Proof. We will construct an isomorphism

$$T : l^q \to (l^p)' \text{ by } Tz = \varphi_z \text{ where } \varphi_z : l^p \to \mathbb{R} \text{ and } \varphi_z(x) = \sum_{i=1}^{\infty} x_i z_i$$

Note that $\varphi_z \in (l^p)'$ since φ_z is trivially a linear functional. So we must show that φ_z is bounded. Use the Holder inequality:

$$|\varphi_z(x)| = \left|\sum_{i=1}^{\infty} x_i z_i\right| \leq \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{\infty}|z_i|^q\right)^{1/q} = \|x\|_p\|z\|_q$$

and therefore $\|\varphi_z\|_{op.} \leq \|z\|_q$.

Next we must show that T is bounded, norm-preserving, and bijective.

"T norm-preserving." We have that $Tz=\varphi_z$ so then $\|Tz\|_{op.} \leq \|z\|_q$ and we want to show equality. See that

$$\|Tz\|_{op.} = \|\varphi_z\|_{op.} = \sup_{x\in l^p, \|x\|=1} |\varphi_z(x)| = \sup_{x\in l^p,\|x\|=1}\left|\sum_{i=1}^{\infty} x_i z_i\right|$$

We want this $\geq \|z\|_q = (\sum_{i=1}^{\infty}|z_i|^q)^{1/q}$. This sup must be bigger than the value at any given x with $\|x\|_p = 1$. So it seems a natural selection for $x \in l^p$ is to take $x_i = \text{sgn}(z_i)\cdot|z_i|^{q-1}$ but we see that

$$\|x\|_p = \left(\sum_{i=1}^{\infty}\left|\text{sgn}(z_i)|z_i)^{q-1}\right|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty}|z_i|^q\right)^{1/p} = \|z\|_q^{q/p} < \infty$$

and thus a better selection so that $\|x\|_p = 1$ is to take $x_i = \frac{\text{sgn}(z_i)|z_i|^{q-1}}{\|z\|_q^{q/p}}$. Therefore we see that

$$\|Tz\|_{op.} \geq \left|\sum_{i=1}^{\infty} x_i z_i\right| = \left|\sum_{i=1}^{\infty}\frac{\text{sgn}(z_i)|z_i|^{q-1}}{\|z\|_q^{q/p}} z_i\right| = \frac{1}{\|z\|_q^{q/p}}\sum_{i=1}^{\infty}|z_i|^q = \frac{\|z\|_q^q}{\|z\|_q^{q/p}} = \|z\|_q$$

This completes the proof.

"T bounded." Trivial as $\|Tz\|_{op.} = \|z\|_q \implies \|T\|_{op.} = 1$.

"T surjective." We want to show that for all $f \in (l^p)'$ there is a $z \in l^q$ such that $f = Tz(= \varphi_z)$. This is the same as showing $f(x) = \varphi_z(x)$ for all $x \in l^p$. Since $f(x) = \sum_{i=1}^{\infty} x_i f(e_i)$ and $\varphi_z(x) = \sum_{i=1}^{\infty} x_i z_i$ where $\{e_i\}$ is the Schauder basis on l^p. Thus it seems a natural selection for z is by $z_i = f(e_i)$. Since $f \in (l^p)'$ we have that it is bounded and thus

$$\left|\sum_{i=1}^{\infty} x_i f(e_i)\right| = |f(x)| \leq \|f\|_{op.}\|x\|_p$$

By the selection of $x_n = \big(\text{sgn}(f(e_1))|f(e_1)|^{q-1}, \text{sgn}(f(e_2))|f(e_2)|^{q-1}, \ldots, \text{sgn}(f(e_n))|f(e_n)|^{q-1}, 0, 0, \ldots\big)$ and $x_n \in l^p$ because

$$\|x_n\|_p = \left(\sum_{i=1}^{n}\left|\text{sgn}(f(e_i))|f(e_i)|^{q-1}\right|^p\right)^{1/p} = \left(\sum_{i=1}^{n}|f(e_i)|^q\right)^{1/p} < \infty$$

and now using the boundedness of f as an operator, we have

$$\sum_{i=1}^{n} |f(e_i)|^q \leq \|f\|_{op.} \cdot \|x\|_p = \|f\|_{op.} \left(\sum_{i=1}^{n} |f(e_i)|^q \right)^{1/p}$$

and therefore

$$\left(\sum_{i=1}^{n} |f(e_i)|^q \right)^{1-1/p} \leq \|f\|_{op.} \implies \|z\|_q = \left(\sum_{i=1}^{n} |f(e_i)|^q \right)^{1/q} \leq \|f\|_{op.} < \infty$$

and therefore $z \in l^q$.

"T injective." Suppose $T(z_1) = T(z_2) \implies T(z_1) - T(z_2) = 0_{map} \implies T(z_1 - z_2) = 0_{map} \implies \|T(z_1 - z_2)\|_{op.} = \|0_{map}\|_{op.}$. Because T is norm preserving, then $\|z_1 - z_2\|_q = \|T(z_1 - z_2)\|_{op.} = \|0_{map}\|_{op.} = \sup_{x \in l^p, x \neq 0} \frac{|0_{map}(x)|}{\|x\|_p} = 0 \implies z_1 - z_2 = 0$ by the definition of a norm and therefore $z_1 = z_2$. Therefore T is injective.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

2. $(l^1)' \cong l^\infty$

Proof. Define an isomorphism

$$T : l^\infty \to (l^1)' \text{ by } Tz = \varphi_z \text{ where } \varphi_z : l^1 \to \mathbb{R} \text{ defined by } \varphi_z(x) = \sum_{i=1}^{\infty} x_i z_i$$

We want to show that T is linear, norm-preserving, injective, and bounded. First we verify that $\varphi_z \in (l^1)'$ by showing it is a bounded linear functional. The linearity and functional parts are trivial. Boundedness follows trivially

$$|\varphi_z(x)| = \left| \sum_{i=1}^{\infty} x_i z_i \right| \leq \sum_{i=1}^{\infty} |x_i||z_i| \leq \sum_{i=1}^{\infty} |x_i| \sup_{i \in \mathbb{N}} |z_i| = \|z\|_\infty \|x\|_1$$

and therefore $\|Tz\|_{op.} = \|\varphi_z\| \leq \|z\|_\infty$ shows that φ_z is bounded and thus in l^1. The fact that T is linear is trivial. Further, the norm-preserving aspect of T verifies boundedness.

"T norm-preserving." We have that $\|Tz\|_{op.} \leq \|z\|_\infty$ so it suffices to show $\|Tz\|_{op.} \geq \|z\|_\infty$ to show equality. See that

$$\|Tz\|_{op.} = \|\varphi_z\|_{op.} = \sup_{x \in l^1, \|x\|=1} |\varphi_z(x)| = \sup_{x \in l^1, \|x\|=1} \left| \sum_{i=1}^{\infty} x_i z_i \right|$$

If $\|z\|_\infty = \sup_{i \in \mathbb{N}} |z_i|$ is actually obtained at z_k then taking $x_i = \delta_{ik} \text{sgn}(z_k)$ it is clear that this is $\geq |z_k| = \|z\|_\infty$. But the sup may not be obtained and thus we can construct a sequence $\{i_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ of components of z such that $z_{i_n} \to \|z\|_\infty$ and $|z_{i_n}| \geq \|z\|_\infty - \frac{1}{n}$. We choose

$$x^{(n)} = (0, \ldots, 0, \underbrace{\text{sgn}(z_{i_n})}_{i_n^{th} \ guy}, 0, \ldots)$$

and therefore the sum with $x^{(n)}$ plugged in for x gives this is $\geq \text{sgn}(z_{i_n}) \cdot z_{i_n} = |z_{i_n}| \to \|z\|_\infty$. Therefore $\geq \|z\|_\infty$ completes this part of the proof.

"T surjective." We want to show that for all $f \in (l^1)'$ there is a $z \in l^\infty$ such that $f = Tz (= \varphi_z)$. But this is the same as saying $f(x) = \varphi_z(x)$ for all $x \in l^1$. But if there is a Schauder basis $\{e_i\}_{i \in \mathbb{N}}$ for l^1 then $f(x) = \sum_{i=1}^{\infty} x_i f(e_i)$ and $\varphi_z(x) = \sum_{i=1}^{\infty} x_i z_i$ indicates a natural selection for z given by $z_i = f(e_i)$. We must show that z defined this way is in l^∞. That is,

we want to show z is a bounded sequence. That is, $|z_i| < M$ for some $M \in \mathbb{R}$ and all $i \in \mathbb{N}$. Since $f \in (l^1)'$, it is bounded and thus for any $x \in l^1$ we have

$$\left|\sum_{i=1}^{\infty} x_i f(e_i)\right| = |f(x)| \leq \|f\|_{op.}\|x\|_1$$

Using this we define a sequence $x^{(n)} = (0,,\ldots,0,\text{sgn} f(e_n),0,\ldots)$ and trivially see that $\|x^{(n)}\|_1 = 1$ and $x^{(n)} \in l^p$ (if $f(e_n) = 0$ for any e_n in the basis, then it would not be a basis element). Thus since the left hand side holds for any $x \in l^p$ we have that for each $n \in \mathbb{N}$,

$$|f(e_n)| \leq \|f\|_{op.} \cdot 1$$

and since $z_n = f(e_n)$ we have shown that $|z_n| \leq \|f\|_{op.}$ for all $n \in \mathbb{N}$ and thus $\sup_{i\in\mathbb{N}} |z_i| \leq \|f\|_{op.} < \infty$ shows $z \in l^\infty$.

"T injective." Suppose $T(z_1) = T(z_2) \implies T(z_1) - T(z_2) = 0_{map} \implies T(z_1 - z_2) = 0_{map} \implies \|T(z_1 - z_2)\|_{op.} = \|0_{map}\|_{op.}$. Because T is norm preserving, then $\|z_1 - z_2\|_1 = \|T(z_1 - z_2)\|_{op.} = \|0_{map}\|_{op.} = \sup_{x\in l^1, x\neq 0} \frac{|0_{map}(x)|}{\|x\|_1} = 0 \implies z_1 - z_2 = 0$ by the definition of a norm and therefore $z_1 = z_2$. Therefore T is injective.

$\mathbb{Q.E.D.}$

3. $(c_0)' \cong l^1$ where $c_0 \subsetneq l^\infty$ is sequences converging to 0 and c_0 is a closed subspace and therefore Banach with the same norm

Proof. c_0 is the space of sequences converging to 0. The dual space of c_0 is $c_0' = \{f : c_0 \to \mathbb{R} \mid f \text{ bounded linear functional}\}$. We want to show that $c_0' \cong l^1$ (i.e. the two are isomorphic). Note that c_0 is a closed subspace of l^∞ and since l^∞ is Banach (complete) and c_0 is closed, then c_0 must also be Banach (complete) by Theorem 1.4-7. Further, we know that norm on c_0 is induced by l^∞ as the sup-norm,

$$\|x\|_{c_0} = \sup_{i\in\mathbb{N}} |x_i|$$

For the rest of the problem we will notate this norm by $\|x\|_\infty$. We want to construct an isomorphism between l^1 and c_0'. Define

$$T : l^1 \to c_0' \text{ by } T(z) = Tz = \varphi_z \text{ where } \varphi_z : c_0 \to \mathbb{R} \text{ defined by } \varphi_z(x) = \sum_{i=1}^{\infty} x_i z_i$$

We first must show that φ_z is a bounded linear functional. It is immediate that it is a functional as the codomain is $\mathbb{R}$.

"φ_z linear." This is immediate as:

- $\varphi_z(x+y) = \sum_{i=1}^\infty (x_i+y_i)z_i = \sum_{i=1}^\infty (x_i z_i + y_i z_i) = \sum_{i=1}^\infty x_i z_i + \sum_{i=1}^\infty y_i z_i = \varphi_z(x) + \varphi_z(y)$
- $\varphi_z(\alpha x) = \sum_{i=1}^\infty (\alpha x_i) z_i = \alpha \sum_{i=1}^\infty x_i z_i = \alpha \varphi_z(x)$

"φ_z bounded." We want to show that $\|\varphi_z\|_{op.} \leq c$ for some constant c. Note that this is equivalent to showing $|\varphi_z(x)| \leq c \cdot \|x\|_{c_0}$ for all $x \in c_0$ by the definition of the operator norm. See that

$$\begin{aligned} |\varphi_z(x)| &= \left|\sum_{i=1}^{\infty} x_i z_i\right| \leq \sum_{i=1}^{\infty} |x_i z_i| = \sum_{i=1}^{\infty} |x_i||z_i| \\ &\leq \sum_{i=1}^{\infty} \left[\left(\sup_{i\in\mathbb{N}} |x_i|\right) \cdot |z_i|\right] = \sum_{i=1}^{\infty} \|x\|_{c_0} \cdot |z_i| \\ &= \|x\|_{c_0} \sum_{i=1}^{\infty} |z_i| = \|x\|_{c_0} \cdot \|z\|_1 \end{aligned}$$

and therefore we have shown that $|\varphi_z(x)| \leq \|z\|_1 \cdot \|x\|_{c_0}$ for all $x \in c_0$ and therefore it trivially follows that $\|\varphi_z\|_{op.} \leq \|z\|_1$.

Now we must show that T is an isomorphism. That is, we need to show that T is linear, bijective, and norm preserving.

"T linear." This is immediate as:

- $T(z_1 + z_2) = \varphi_{z_1+z_2}$. But then for $x \in c_0$,

$$\begin{aligned}\varphi_{z_1+z_2}(x) &= \sum_{i=1}^{\infty} x_i(z_1+z_2)_i = \sum_{i=1}^{\infty} x_i\left[z_i^{(1)} + z_i^{(2)}\right] = \sum_{i=1}^{\infty}\left[x_i z_i^{(1)} + x_i z_i^{(2)}\right] \\ &= \sum_{i=1}^{\infty} x_i z_i^{(1)} + \sum_{i=1}^{\infty} x_i z_i^{(2)} = \sum_{i=1}^{\infty} x_i(z_1)_i + \sum_{i=1}^{\infty} x_i(z_2)_i \\ &= \varphi_{z_1}(x) + \varphi_{z_2}(x) = \left(\varphi_{z_1} + \varphi_{z_2}\right)(x)\end{aligned}$$

  and therefore $\varphi_{z_1+z_2}(x) = \left(\varphi_{z_1} + \varphi_{z_2}\right)(x)$ for all $x \in c_0$ and therefore they must be the same map. That is, $\varphi_{z_1+z_2} = \varphi_{z_1} + \varphi_{z_2}$.
- $T(\alpha z) = \varphi_{\alpha z}$. But then for $x \in c_0$,

$$\varphi_{\alpha z}(x) = \sum_{i=1}^{\infty} x_i(\alpha z)_i = \sum_{i=1}^{\infty} x_i \alpha z_i = \alpha \sum_{i=1}^{\infty} x_i z_i = \alpha\varphi_z(x) = (\alpha\varphi_z)(x)$$

  and since $\varphi_{\alpha z}(x) = (\alpha\varphi_z)(x)$ for all $x \in c_0$, then they are the same map and thus $\varphi_{\alpha z} = \alpha\varphi_z$.

"T norm preserving." We want to show that $\|Tz\|_{op.} = \|z\|_1$ for all $z \in l^1$. For $z = 0$, by the linearity of T, $Tz = 0$ map $\implies \|Tz\|_{op.} = 0$ and also $\|z\|_1 = 0$ by positive-definiteness. Therefore when $z = 0$ clearly this is satisfied. Thus assume $z \neq 0$, $z \in l^1$. Note from the boundedness of φ_z we showed that $\|\varphi_z\|_{op.} \leq \|z\|_1$ and since $Tz = \varphi_z$, this shows that $\|Tz\|_{op.} \leq \|z\|_1$. See that

$$\|Tz\|_{op.} = \|\varphi_z\|_{op.} = \sup_{x\in c_0, \|x\|_\infty = 1} |\varphi_z(x)| = \sup_{x\in c_0, \|x\|_\infty = 1} \left|\sum_{i=1}^{\infty} x_i z_i\right|$$

and choose $x_n \in c_0$ by $x_n = (\text{sgn}(z_1), \text{sgn}(z_2), \ldots, \text{sgn}(z_n), 0, 0, \ldots)$. Since $z \neq 0$, then at least one component is non-zero. That is, $\exists\ N \in \mathbb{N}$ such that $z_N \neq 0 \implies |\text{sgn}(z_N)| = 1$ and thus for $n \geq N$, $\|x_n\|_\infty = \sup_{i\in\mathbb{N}} |x_i^{(n)}| = \sup_{i\in\mathbb{N}} |\text{sgn}(z_i)| = 1$. Therefore each x_n for $n \geq N$ satisfies the criteria for taking the sup and thus

$$\|Tz\|_{op.} = \sup_{x\in c_0, \|x\|_\infty = 1} \left|\sum_{i=1}^{\infty} x_i z_i\right| \geq \left|\sum_{i=1}^{\infty} x_i^{(n)} z_i\right| = \left|\sum_{i=1}^{n} \text{sgn}(z_n) z_i\right| = \sum_{i=1}^{n} |z_i| \quad \forall \quad n \geq N$$

and therefore

$$\|Tz\|_{op.} \geq \sum_{i=1}^{\infty} |z_i| = \|z\|_1$$

Thus we have shown that $\|Tz\|_{op.} = \|z\|_1$ by showing that $\|Tz\|_{op.} \leq \|z\|_1$ and $\|Tz\|_{op.} \geq \|z\|_1$.

"T injective." Suppose $T(z_1) = T(z_2) \implies T(z_1) - T(z_2) = 0_{map} \implies T(z_1 - z_2) = 0_{map} \implies \|T(z_1 - z_2)\|_{op.} = \|0_{map}\|_{op.}$. Because T is norm preserving, then $\|z_1 - z_2\|_1 = \|T(z_1 - z_2)\|_{op.} = \|0_{map}\|_{op.} = \sup_{x\in c_0, x\neq 0} \frac{|0_{map}(x)|}{\|x\|_\infty} = 0 \implies z_1 - z_2 = 0$ by the definition of a norm and therefore $z_1 = z_2$. Therefore T is injective.

"T surjective." We want to show that $\forall\ f \in c_0'\ \exists\ z \in l^1$ such that $Tz = f$. But note that $Tz = \varphi_z$ and thus we want to show that $\varphi_z = f$. But this simply means that we want to show that $\varphi_z(x) = f(x)$ for all $x \in c_0$. But note that if we have a Schauder basis on c_0, then we can write $f(x) = \sum_{i=1}^{\infty} x_i f(e_i)$ and we knew a priori that $\varphi_z(x) = \sum_{i=1}^{\infty} x_i z_i$. Therefore,

we see the natural selection of $z_i = f(e_i)$ to satisfy this surjectivity. Therefore we must show the following: c_0 has a Schauder basis, construct a Schauder basis and show that any $x \in c_0$ can be written as infinite sum of this Schauder basis' elements, and show that $z \in l^1$ by our definition.

"c_0 has S. basis & construction of S. basis." Define

$$e_i = (0, 0, \ldots, 0, 0, \underbrace{\quad 1 \quad}_{i^{th} \text{ component}}, 0, 0, \ldots)$$

which is clearly in c_0 by construction. Therefore, $\{e_i\}_{i \in \mathbb{N}} \subseteq c_0$. In order to show this is a Schauder basis for c_0, we must show that $\forall\ x \in c_0\ \exists!\ \{x_i\} \subseteq \mathbb{R}$ such that $x = \sum_{i=1}^{\infty} x_i e_i$. That is, $\sum_{i=1}^{n} x_i e_i \to x$ as $n \uparrow \infty$. This is easy to show as:

$$\begin{aligned} \left\| \sum_{i=1}^{n} x_i e_i - x \right\| &= \|(x_1, x_2, \ldots, x_n, 0, 0, \ldots) - (x_1, x_2, \ldots)\| \\ &= \|(0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)\| = \sup_{i \geq n+1} |x_i| \end{aligned}$$

which converges to 0 as $n \uparrow \infty$ since $x \in c_0$. Then $\|\sum_{i=1}^{n} x_i e_i - x\| \to 0$ as $n \uparrow \infty$ and thus $\sum_{i=1}^{n} x_i e_i \to x$ as $n \uparrow \infty$. Therefore, each $x \in c_0$ can be written as an infinite combination of this Schauder basis we have constructed.

"$z \in l^1$." We naturally define z by $z_i = f(e_i)$ where e_i is defined as above. We want to show that $z \in l^1$. That is, we want to show that $\|z\|_1 < \infty$ which is the same as showing $\sum_{i=1}^{\infty} |f(e_i)| < \infty$. Note that since $f \in c_0'$, then f is a bounded linear functional and therefore

$$\left| \sum_{i=1}^{\infty} x_i f(e_i) \right| = |f(x)| \leq \|f\|_{op.} \cdot \|x\|_{\infty} \quad \forall\ x \in c_0$$

Since this holds for all $x \in c_0$, if we choose $x_n = (\text{sgn}(f(e_1)), \text{sgn}(f(e_2)), \ldots, \text{sgn}(f(e_n)), 0, 0, \ldots)$, then clearly $x_n \in c_0$ and further $\|x_n\|_{\infty} = 1$. Then

$$\left| \sum_{i=1}^{\infty} x_i f(e_i) \right| \geq \left| \sum_{i=1}^{\infty} x_i^{(n)} f(e_i) \right| = \left| \sum_{i=1}^{n} \text{sgn}(f(e_i)) f(e_i) \right| = \sum_{i=1}^{n} |f(e_i)|$$

and then we have that

$$\sum_{i=1}^{n} |f(e_i)| \leq \left| \sum_{i=1}^{\infty} x_i f(e_i) \right| \leq \|f\|_{op.} \cdot 1 \qquad \forall\ n \in \mathbb{N}$$

and thus

$$\sum_{i=1}^{\infty} |f(e_i)| \leq \|f\|_{op.} < \infty \text{ since } f \in c_0'$$

Therefore we have shown what we wanted and thus $z \in l^1$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Section 3.1. Inner Product Space. Hilbert Space

Inner product space/inner product. X is an inner product space if X is a normed vector space with norm induced from an inner product. An inner product satisfies $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$

1. Bilinear (with respect to conjugacy). That is,

$$\begin{aligned}\langle \alpha x_1 + \alpha x_2, y\rangle &= \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle \\ \langle x, \alpha y_1 + \beta y_2\rangle &= \bar{\alpha}\langle x, y_1\rangle + \bar{\beta}\langle x, y_2\rangle\end{aligned}$$

2. Conjugate-symmetric
$$\langle x, y\rangle = \overline{\langle y, x\rangle}$$

3. Positive-definite
$$\langle x, x\rangle \geq 0 \text{ and } \langle x, x\rangle = 0 \iff x = 0$$

Norm induced by inner product. $\|x\| = \sqrt{\langle x, x\rangle}$

Property. Any norm induced from an inner product satisfies $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Not an inner product space. $\mathcal{C}[a,b]$ with $\|f\| = \sup_{t\in[a,b]} |f(t)|$ needs to satisfy $\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$. Can construct functions making this false.

Hilbert space. Complete inner product space.

Orthogonal. $x \perp y \iff \langle x, y\rangle = 0$

Section 3.2. Further Properties of Inner Product Spaces

Cauchy-Schwartz inequality. $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ and equality holds only if $y = c \cdot x$ for some $c \in \mathbb{R}$.

Proof. See that
$$\langle x + \alpha y, x + \alpha y\rangle = \|x\|^2 + \bar{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} + |\alpha|^2\|y\|^2$$
for any $\alpha \in \mathbb{K}$. By positive-definiteness we have that this quantity must be non-negative. Choose $\alpha = t \cdot \langle x, y\rangle$ and thus this become
$$= \|x\|^2 + 2t|\langle x, y\rangle|^2 + t^2|\langle x, y\rangle|^2\|y\|^2$$
which is quadratic in t. Since this quantity is non-negative then there are 0 or 1 roots and so we have the coefficients $b^2 - 4ac \leq 0$. Thus,

$$4t^2|\langle x, y\rangle|^4 - 4\|x\|^2 t^2 |\langle x, y\rangle|^2\|y\|^2 \leq 0 \iff 4t^2|\langle x, y\rangle|^2\left(|\langle x, y\rangle|^2 - \|x\|^2\|y\|^2\right) \leq 0 \iff |\langle x, y\rangle|^2 - \|x\|^2\|y\|^2 \leq 0$$

and the inequality immediately follows.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Continuity of inner product. $x_n \to x$ and $y_n \to y \implies \langle x_n, y_n\rangle \to \langle x, y\rangle$.

Proof. See that

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &= |\langle x_n, y_n - y\rangle - (\langle x_n - x, y\rangle)| \\ &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \underbrace{\|x_n\|}_{\text{bounded b/c } x_n \text{ conv.}} \|y_n - y\| + \|x_n - x\| \underbrace{\|y\|}_{\text{fixed}} \\ &\to 0 \text{ as } n \uparrow \infty\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Completion of Inner Product Spaces

Completion of metric spaces. Recall X is a metric space $\implies \exists!\ \hat{X}$ complete metric space such that $\exists\ W \subseteq \hat{X}$ dense and $W \cong X$ (isometric, i.e. $\exists\ T : W \to X$ isometric (bijective, metric preserving)).

Theorem for inner products. X is an inner product space $\implies \exists!\ H$ Hilbert space such that $\exists\ W \subseteq H$ and $W \cong X$ (isomorphic, i.e. $\exists\ T : W \xrightarrow[linear]{bij.} X$ that preserves inner product).

Proof. Define $\langle \underbrace{\hat{x}}_{=[\{x_n\}]}, \underbrace{\hat{y}}_{=[\{y_n\}]} \rangle_H = \lim_{n\to\infty}\langle x_n, y_n\rangle$ on $H = \{\hat{x} = [\{x_n\}] \mid \{x_n\} \text{ Cauchy in } X\}$ with equivalency classes $[\{x_n\}]$ structured by equivalence relation $\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) = 0$ where d induced by norm induced by inner product. We must show this.

We must show that 1) $\langle \cdot,\cdot\rangle_H$ is well-defined, 2) the limit exists, 3) it defines an inner product, and 4) $\langle \cdot,\cdot\rangle_H$ induces $\hat{d}$.

1. Suppose $\{x_n\}, \{x_n'\} \in \hat{x}$ and $\{y_n\}, \{y_n'\} \in \hat{y}$. Note a prior that $\{x_n\} \sim \{x_n'\}$ and $\{y_n\} \sim \{y_n'\}$. We WTS $\lim_{n\to\infty}\langle x_n, y_n\rangle = \lim_{n\to\infty}\langle x_n', y_n'\rangle$. See that

$$|\langle x_n, y_n\rangle - \langle x_n', y_n'\rangle| \le \|x_n - x_n'\| \cdot \|y_n'\| + \|x_n'\| \cdot \|y_n - y_n'\| \to 0$$

   since both $\|y_n'\|$ and $\|x_n'\|$ are bounded (since $\{x_n'\}, \{y_n'\}$ converge).

2. Note that $\langle x_n, y_n\rangle$ is a sequence in $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$, both complete) and thus if it is Cauchy then it converges. We'll show it is Cauchy. See that

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| &= |\langle x_n - x_m, y_n\rangle + \langle x_m, y_n - y_m\rangle| \le |\langle x_n - x_m, y_n\rangle| + |\langle x_m, y_n - y_m\rangle| \\ &\le \|x_n - x_m\| \cdot \|y_n\| + \|x_n\| \cdot \|y_n - y_m\| \to 0 \end{aligned}$$

   since $\{\|y_n\|\}, \{\|x_n\|\}$ are both bounded sequences.

3. Only difficult thing to check is positive definiteness:

$$\hat{x} = 0 \iff \{x_n\} \sim \{(0,0,\ldots)\} \iff \lim_{n\to\infty} d(x_n, 0) = 0 \iff \lim_{n\to\infty}\langle x_n, x_n\rangle = 0 \iff \lim_{n\to\infty}\langle \hat{x}, \hat{x}\rangle_H = 0$$

4. Does this inner product induce $\hat{d}$?

$$\begin{aligned} d_{\langle\cdot,\cdot\rangle_H}(\hat{x}, \hat{y}) &= \|\hat{x} - \hat{y}\| = \sqrt{\langle \hat{x}, \hat{y}\rangle_H} = \sqrt{\lim_{n\to\infty}\langle x_n - y_n, x_n - y_n\rangle} = \lim_{n\to\infty}\sqrt{\langle x_n - y_n, x_n - y_n\rangle} \\ &= \lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty} d(x_n, y_n) = \hat{d}(\hat{x}, \hat{y}) \end{aligned}$$

Last we need to show that there is an isomorphism $T : X \to W \subseteq H$. Construct it by $Tx = [(x, x, \ldots)]$. Bijective? by metric space completion. Linear? by metric space completion. Need to check norm preserving, easy:

$$\langle Tx, Ty\rangle_H = \lim_{n\to\infty}\langle x, y\rangle_X = \langle x, y\rangle_X$$

$\mathbb{Q.E.D.}$

Theorem (subspace). Let Y be a subspace of a Hilbert space H. Then:

- Y complete $\iff Y$ closed in H
- $\dim Y < \infty \implies Y$ complete
- H separable $\implies Y$ separable

Section 3.3. Orthogonal Complements and Direct Sums

Optimization theorem. Let X be an inner product space and $M \subseteq X$ closed and complete. Then $\forall\, x \in X,\ \exists!\ y \in M$ such that $d(x, M) = d(x, y)$.

Proof. Let $x \in X$ and $\delta = d(x, M) = \inf_{z \in M} d(x, z)$. If $\delta = 0$ then trivial because then we would have a sequence $\{z_n\} \subseteq M$ such that $z_n \to y$ with y satisfying $d(x, y) = 0$. But then $y \in M$ because M closed.

Assume $\delta > 0$. Then $\exists\ \{y_n\} \subseteq M$ such that $d(x, y_n) \to \delta$ as $n \uparrow \infty$. We WTS $\{y_n\}$ is Cauchy (and since M is complete, then $y_n \to y \in Y$). Since X is an inner product space we have for $A, B \in X$

$$\|A + B\|^2 + \|A - B\|^2 = 2\left(\|A\|^2 + \|B\|^2\right)$$

and taking $A = x - y_n$ and $B = x - y_m$. (Note that trivially $\|x - y_n\| \to \delta$ and $\|x - y_m\| \to \delta$.) Then

$$\|y_n - y_m\|^2 + 4\left\|x - \frac{y_n + y_m}{2}\right\| = 2\left(\|x - y_n\|^2 + \|x - y_m\|^2\right)$$

and thus

$$\|y_n - y_m\|^2 = 2\left(\|x - y_n\|^2 + \|x - y_m\|^2\right) - 4\left\|x - \frac{y_n + y_m}{2}\right\|$$

Since $\|x - y_n\| \to \delta$, the

$$\forall\, \epsilon > 0,\ \exists\ N_1 \in \mathbb{N} \quad \text{such that} \quad \left|\|x - y_n\|^2 - \delta^2\right| < \frac{\epsilon}{8} \text{ if } n \geq N_1$$
$$\implies \quad \|x - y_n\|^2 < \delta^2 + \frac{\epsilon}{8} \text{ if } n \geq N_1$$

Noting that $\frac{y_n + y_m}{2}$ is in M since it is a convex combination of two elements of M and M is convex, then

$$\left\|x - \frac{y_n + y_m}{2}\right\| = d\left(x, \frac{y_n + y_m}{2}\right) \geq \inf_{z \in M} d(x, M) = \delta \implies -4\left\|x - \frac{y_n + y_m}{2}\right\| \leq -4\delta^2$$

Thus,

$$\begin{aligned} \|y_n - y_m\|^2 &\leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\delta^2 \\ &< 2\left(\frac{\epsilon}{8} + \delta^2\right) + 2\left(\frac{\epsilon}{8} + \delta^2\right) - 4\delta^2 \\ &= \frac{\epsilon}{2} < \epsilon \text{ if } n, m \geq N_1 \end{aligned}$$

Therefore $\{y_n\}$ is Cauchy and converges to a $y \in M$.

Uniqueness? Assume that $\exists\ y_1, y_2 \in M$ such that $d(x, y_1) = d(x, y_2) = \delta$. By the paralellogram identity,

$$\|y_1 - y_2\| + 4\left\|x - \frac{y_n + y_m}{2}\right\| = 2\left(\|x - y_n\|^2 + \|x - y_m\|^2\right) \implies \|y_1 - y_2\|^2 = 4\delta^2 - 4\left\|x - \frac{y_1 + y_2}{2}\right\|^2 \leq 4\delta^2 - 4\delta^2 = 0$$

$\mathbb{Q.E.D.}$

Corollary. $Y \subseteq X$ is complete subspace by the above gives us $\forall\, x \in X,\ \exists!\ y \in Y$ such that $\|x - y\| = d(x, Y)$. Then $x - y \perp Y$.

Proof. Assume for contradiction that $x - y \not\perp Y$. That is, $\exists\ y_1 \in Y$ such that $\langle x - y, y_1 \rangle \neq 0$. Let $u = x - y$. Then $\langle u, u \rangle = \|x - y\|^2$. Note that since y was the mimizer for the distance between x and M that if we can find a $z \in Y$ such that $\|x - z\|^2 < \|x - y\|^2$ we have a contradiction. We take a $z \in Y$ of the form $y + \alpha y_1$ for some $\alpha \in \mathbb{K}$. Then

$$\|x - (y + \alpha y_1)\|^2 = \|u - \alpha y_1\|^2 = \langle u - \alpha y_1, u - \alpha y_1 \rangle = \|u\|^2 - \bar{\alpha}\langle u, y_1 \rangle - \alpha\overline{\langle u, y_1 \rangle} + |\alpha|^2\|y_1\|^2$$

and if we take $\alpha = \frac{\langle u, y_1 \rangle}{\|y_1\|^2} \implies \bar{\alpha} = \overline{\frac{\langle u, y_1 \rangle}{\|y_1\|^2}}$ then the above is

$$\begin{aligned} &= \|x-y\|^2 - \underbrace{\frac{|\langle u, y_1\rangle|^2}{\|y_1\|^2}}_{>0 \ by \ hyp.} \\ &< \|x-y\|^2 \end{aligned}$$

giving a contradiction.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Direct sum corollary. If H is Hilbert, then $Y \subseteq H$ closed subspace ($\implies$ complete) $\implies H = Y \oplus Y^{\perp}$ where $Y^{\perp}$ = othogonal complement of $Y = \{z \in H \mid z \perp Y\}$.

Claim. Such a decomposition of any element in H is unique.

Theorem. Y is a closed subspace of a Hilbert space $H \iff Y = Y^{\perp\perp}$.

Proof. " $\implies$ " Suppose Y is a closed in H. See that $Y \subseteq Y^{\perp\perp}$ because $y \in Y \implies y \perp Y^{\perp} \implies y \in (Y^{\perp})^{\perp}$. Thus we will show $Y \supseteq Y^{\perp\perp}$. Let $x \in Y^{\perp\perp}$. Then since $x \in H$ we have by Theorem 3.4-4 that $x = y + z$ for $y \in Y \subseteq Y^{\perp\perp}$ and for some $z \in Y^{\perp}$ (since $H = Y \oplus Y^{\perp}$). Since $Y^{\perp\perp}$ is a vector space and $x \in Y^{\perp\perp}$ then $z = x - y \in Y^{\perp\perp}$ since both x and y are in $Y^{\perp\perp}$ and thus using previously that $z \in Y^{\perp}$, we must have that $z \perp z \implies \langle z, z\rangle = 0 \implies z = 0$ by the positive-definiteness of the inner product on H. Then $x = y \implies x \in Y$. Thus $Y \supseteq Y^{\perp\perp}$ and therefore $Y = Y^{\perp\perp}$.

"$\impliedby$" Suppose $Y = Y^{\perp\perp}$. We will use Theorem 3.2-4, that a subspace Y of H is complete if and only if it is closed in H. Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$ is a Cauchy sequence in Y. Then it is a Cauchy sequence in H since $Y \subseteq H$ and therefore it converges. Thus $x_n \to x \in H$. But since $\{x_n\}_{n\in\mathbb{N}} \subseteq Y = Y^{\perp\perp}$, then $x_n \perp Y^{\perp} \implies \langle x_n, y\rangle = 0$ for all $n \in \mathbb{N}$ and $y \in Y^{\perp}$. We want to show that $x \perp Y^{\perp}$, which would directly imply that $x \in Y^{\perp\perp} = Y$ and show the completeness of Y. See that for arbitrary $y \in Y^{\perp}$,

$$\langle x, y\rangle = \left\langle \lim_{n\to\infty} x_n, y \right\rangle \underbrace{=}_{cont. \ of \ in. \ pd.} \lim_{n\to\infty} \langle x_n, y\rangle = \lim_{n\to\infty} 0 = 0$$

This shows that $x \perp Y^{\perp} \implies x \in Y^{\perp\perp} = Y$. Therefore, any Cauchy sequence in Y converges in Y and thus Y is complete. Since it is a subspace of a Hilbert space then it must be closed.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Lemma. Let $M \subseteq H$ be nonempty and H be Hilbert. $\overline{\text{span}M} = H \iff M^{\perp} = \{0\}$.

Proof. Suppose $M \subseteq H$ is nonempty and H is Hilbert.

" $\implies$ " Assume $\overline{\text{span}M} = H$. Let $x \in M^{\perp}$ and since $M^{\perp} \subseteq H = \overline{\text{span}M} \implies \exists \{y_n\} \subseteq \text{span}M$ such that $y_n \to x$.

$$\langle x, x\rangle = \lim_{n\to\infty} \langle x, y_n\rangle = \lim_{n\to\infty} \left\langle x, \sum_{i=1}^{\dim M} \alpha_i^{(n)} m_i \right\rangle = \lim_{n\to\infty} \sum_{i=1}^{\dim M} \overline{\alpha_i^{(n)}} \underbrace{\langle x, m_i\rangle}_{=0} = 0$$

and therefore $x = 0 \implies M^{\perp} = \{0\}$.

"$\impliedby$" Let $Y = \overline{\text{span}M} \subseteq H$ which is a closed subspace. Then $H = Y \oplus Y^{\perp} = \left(\overline{\text{span}M}\right) \oplus \left(\overline{\text{span}M}\right)^{\perp}$. Then $x \in H$ can be written as $x = y + z$ where $y \in Y$ and $z \in Y^{\perp}$. We want to show that $z = 0$ in order to show that $x = y \in Y \implies x \in Y$ and then $H \subseteq Y$. See that $M \subseteq Y \implies Y^{\perp} \subseteq M^{\perp} = \{0\}$ and thus $z = 0$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Section 3.4. Orthonormal Sets and Sequences

Orthogonal set. $\{x_\alpha\}_{\alpha\in I}$ is orthogonal $\iff x_\alpha \perp x_\beta$ for all $\alpha, \beta \in I, \alpha \neq \beta$

Orthonormal set. $\{x_\alpha\}_{\alpha\in I}$ is orthonormal $\iff x_\alpha \perp x_\beta$ for all $\alpha, \beta \in I, \alpha \neq \beta$ and $\langle x_\alpha, x_\beta\rangle = \delta_{\alpha\beta}$.

Pythagorean relation. If x and y are orthonormal elements then trivially $\langle x, y\rangle = 0$ and further $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Lemma (linear independence). An orthonormal set is linearly independent.

Proof. Consider

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$$

and then take $\langle \sum_k \alpha_k e_k, e_j\rangle = \sum_k \alpha_k \langle e_k, e_j\rangle = \alpha_j = 0$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Representation of elements. If $\{e_i\}_{i=1,\ldots,n}$ is an orthonormal set in X then for any $x \in X$ we already knew we could write X as a linear combination of these elements, but we further obtain

$$x = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$$

Bessel's inequality. For any $x \in X$,

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$$

Proof. If $y \in Y_n \implies x - y \perp y$ and thus

$$\|x\|^2 = \|y\|^2 + \|x-y\|^2$$

and using $y = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$. $Y_n = \text{span}\{e_1, \ldots, e_n\}$.

Gram-Schmidt Process

Can we construct an orthonormal set from a linearly independent set? Let $\{x_i\}_{i=1,\ldots,n}$ be linearly independent.

$$\begin{aligned}
e_1 &= \frac{x_1}{\|x_1\|} \\
e_2 &= \frac{x_2 - \overbrace{\langle x_2, e_1\rangle e_1}^{P_{sp(x_1)} x_2}}{\|x_2 - \langle x_2, e_1\rangle e_1\|} \\
&\vdots \\
e_k &= \frac{x_k - \sum_{i=1}^{k-1} \langle x_k, e_i\rangle e_i}{\left\| x_k - \sum_{i=1}^{k-1} \langle x_k, e_i\rangle e_i \right\|} \\
&\vdots
\end{aligned}$$