MA 515
Final Study Guide

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Section 3.5. Series Related to Orthonormal Sequences.

Theorem (Convergence). Let (e_k) be an orthonormal sequene in a Hilbert space H. Then:

(a) $\sum_{k=1}^{\infty} \alpha_k e_k$ converges (in the norm on H) $\iff \sum_{k=1}^{\infty} |\alpha_k|^2$ converges.

(b) $x = \sum_{k=1}^{\infty} \alpha_k e_k$ converges $\implies \alpha_k = |\langle x, e_k \rangle|$.

(c) $x \in H$, $x = \sum_{k=1}^{\infty} \alpha_k e_k$ converges with $\alpha_k = \langle x, e_k \rangle$ converges.

Proof. (a) Let $s_n = \sum_{i=1}^{n} \alpha_i e_i$ and note

$$\begin{aligned} \|s_m - s_n\|^2 &= \left\| \sum_{i=n+1}^{m} \alpha_i e_i \right\|^2 = \sum_{i=n+1}^{m} \|\alpha_i e_i\|^2 \quad (\textit{Pythagorean theorem}) \\ &= \sum_{i=n+1}^{m} |\alpha_i|^2 = t_m - t_n \end{aligned}$$

if we take $t_k = \sum_{i=1}^{k} |\alpha_i|^2$. Thus if one converges then the other must.

(b) $x = \sum_{i=1}^{\infty} \alpha_i e_i$ exists $\iff s_n = \sum_{i=1}^{n} \alpha_i e_i \to x$. Note

$$\langle x, e_i \rangle = \left\langle \lim_{n\to\infty} s_n, e_i \right\rangle = \lim_{n\to\infty, n>i} \langle s_n, e_i \rangle = \lim_{n\to\infty, n>i} \left\langle \sum_{j=1}^{n} \alpha_j e_j, e_i \right\rangle$$

which is 0 if $n \leq i$ so assume that $n > i \implies$ every term is 0 except for i^{th} one $= \alpha_i$. Thus $\alpha_i = \langle x, e_i \rangle$ for all $i \in \mathbb{N}$.

(c) By (a) we have that $\sum_{i=1}^{\infty} |\langle \alpha, e_i \rangle|^2$ exists by Bessel Inequality.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Lemma (Fourier coefficients). Any x in X inner product space can have at most countably many nonzero Fourier coefficients $\langle x, e_k \rangle$ with respect to an orthonormal family $(e_k), k \in I$, in X.

Proof. Let $x \in H$ and write $x = \sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha$ for all $x \in X$ which is an uncountable sum. But if we can show that there are a countable number of non-zero Fourier coefficients. Define the set for fixed $x \in H$, $J_x = \{\alpha \in I \mid \langle x, e_\alpha \rangle \neq 0\} \subseteq I$. We may then write

$$x = \sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha = \sum_{\alpha \in J_x} \langle x, e_\alpha \rangle e_\alpha$$

We want to show that the set J_x is countable. Define

$$J_k = \left\{ \alpha \in I \middle| \langle x, e_\alpha \rangle > \frac{1}{k} \right\}$$

noting $J_k \subseteq J_{k+1}$ and defining $J = \bigcup_{k=1}^{\infty} J_k = \lim_{k\to\infty} J_k$. We want to show that each J_k is countable in order to show that J is countable (as we would obtain a countable union of countable sets). Choose $M \subseteq J_k$ such that $M = \{\alpha_1, \ldots, \alpha_m\} \subseteq J_k$ is a finite set. Then since $\langle x, e_\alpha \rangle > \frac{1}{k}$ we then have

$$m \cdot \frac{1}{k^2} < \sum_{i=1}^{m} |\langle x, e_\alpha \rangle|^2 \leq \|x\|^2 < \infty$$

and noting that the LHS $\uparrow \infty$ as $m \uparrow \infty$ gives a contradiction and thus m must be fixed a priori and thus each J_k must be finite, showing the countability of J_x.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Section 3.6. Total Orthonormal Sets and Sequences.

Total orthonormal set. A total set in a normed space X is a subset $M \subseteq X$ whose span is dense in X. Accordingly, an orthonormal set in an inner product space X which is total in X is called a *total orthonormal set* in X. That is, M is total in $X \iff \overline{\text{span}M} = X$.

Theorem (totality). Let M be a subset of an inner product space X. Then:

(a) If M is total in X, then there does not exist a nonzero $x \in X$ whch is orthogonal to every element of M; that is, $x \perp M \implies x = 0$.

(b) If X Hilbert, then $x \perp M \implies x = 0$ shows M total in X.

Facts. (a) M is total in Hilbert $H \iff M^\perp = \{0\}$

(b) M total in $H \iff \overline{\text{span}M} = H$

(c) M total $\iff$ Parseval's equality holds, i.e. $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2$

Theorem (Separable Hilbert spaces). Let H be a Hilbert space. Then:

(a) If H separable, every orthonormal set in H is countable.

(b) If H contains an orthonormal sequence which is total in H, then H is separable.

Section 3.7. Lenendre, Hermite and Laguerre Polynomials.

Legendre polynomials. Can represent them in many ways:

$$\begin{aligned} P_n(t) &= \frac{1}{2^n n!} \frac{d^n}{dt^n} [(t^2-1)^n] \\ &= \sum_{j=0}^{N} (-1)^j \frac{(2n-2j)!}{2^n j!(n-j)!(n-2j)!} t^{n-2j} \quad , \quad N = \frac{n}{2} \text{ or } \frac{n-1}{2} \text{ if even/odd} \end{aligned}$$

First few polynomials given by:

$$\begin{aligned}
P_0(t) &= 1 \\
P_1(t) &= t \\
P_2(t) &= \frac{1}{2}(3t^2 - 1) \\
P_3(t) &= \frac{1}{2}(5t^3 - 3t) \\
P_4(t) &= \frac{1}{8}(35t^4 - 30t^2 + 3) \\
&\vdots
\end{aligned}$$

And applying G-S process we can arrive at

$$e_n = \sqrt{\frac{2n+1}{2}} P_n(t)$$

Section 3.8. Representation of Functionals on Hilbert Spaces.

Riesz Lemma. Y is a closed subspace of normed vector space $X \implies \forall\ \theta \in (0,1)\ \exists\ x \in S_X(0,1)$ such that $d(x,Y) > \theta$.

Riesz's Theorem (RR Thm baby). Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely, $f(x) = \langle x, z\rangle$ where z depends on f, is uniquely determined by f and has norm $\|z\|_H = \|f\|_{op}$.

Proof. See that for $f \in H'$ we have $f(f(x)\cdot a - f(a)\cdot x)$=0 for all $a, x \in H$ trivially. Then $f(x)a - f(a)x \in N = \ker f$. N is a closed subspace of H and thus $H = N \oplus N^\perp$. If $N^\perp = \{0\}$ then $H = N = \ker f \implies f \equiv 0$ so choose $z = 0$ for the inner product.

If $N^\perp \supsetneq \{0\}$ then $\exists\ a \in N$ with $a \neq 0$ such that $\langle \underbrace{f(x)a - f(x)x}_{\in N}, \underbrace{a}_{\in N^\perp}\rangle = 0 \implies f(x)\|a\|^2 = f(a)\langle x, a\rangle \implies$

$f(x) = \frac{f(a)}{\|a\|^2}\langle x, a\rangle = \left\langle x, \underbrace{\frac{\overline{f(a)}}{\|a\|^2}\cdot a}_{=z}\right\rangle$

ℚ.𝔼.𝔻.

Definition (Sesquilinear form). Let X and Y be vector spaces over the same field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Then a *sesquilinear form* (or *sesquilinear functional*) h on $X \times Y$ is a mapping $h : X \times Y \to \mathbb{K}$ such that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ we have

$$\begin{aligned}
h(x_1 + x_2, y) &= h(x_1, y) + h(x_2, y) \\
h(x, y_1 + y_2) &= h(x, y_1) + h(x, y_2) \\
h(\alpha x, y) &= \alpha h(x, y) \\
h(x, \beta y) &= \bar{\beta} h(x, y)
\end{aligned}$$

Note if $\mathbb{K} = \mathbb{R}$ then the last condition simply gives this is a *bilinear form*.

Norm on h. h is bounded if $|h(x,y)| \leq c\|x\|\|y\|$ for some $c \in [0, \infty)$. The norm is given by

$$\|h\| = \sup_{x \in X - \{0\}, y \in Y - \{0\}} \frac{|h(x,y)|}{\|x\|\|y\|} = \sup_{\|x\|=1, \|y\|=1} |h(x,y)|$$

Theorem (Riesz representation adult). Let H_1, H_2 be Hilbert spaces and $h : H_1 \times H_2 \to \mathbb{K}$ a *bounded sesquilinear form*. The h has representation $h(x, y) = \langle Sx, y\rangle$ where $S : H_1 \to H_2$ is a bounded linear operator. S is uniquely determined by h and has norm $\|S\| = \|h\|$.

Proof. Fix $x \in H_1$ and let $f_x : H_2 \to \mathbb{K}$ defined by $f_x(y) = \overline{h(x, y)}$ which is clearly bounded and linear. Bounded because $\|f\|_{op} \leq \|h\|_{sesq}\|x\|_H$. Thus by RR Theorem (baby) we have

$$\exists!\ z_x \in H_2 \text{ s.t. } f_x(\cdot) = \langle \cdot, z_x\rangle_{H_2}$$

but we have that $f_x(\cdot) = \overline{h(x, \cdot)} \implies \overline{h(x, \cdot)} = \langle \cdot, z_x\rangle_{H_2} \implies h(x, \cdot) = \langle z_x, \cdot\rangle_{H_2}$. Thus for any choice of x we may form this relationship between h and the inner product with choice of z_x.

Define $S : H_1 \to H_2$ by $Sx = z_x$. By construction we trivially have that $h(x, y) = \langle z_x, y\rangle = \langle Sx, y\rangle$ for all $y \in H_2$ for fixed $x \in H_1$. Linearity is easy to show. Must show bounded operator and norm-preserving:

"Bounded." WTS $\|Sx\|_{H_2} \leq c \cdot \|x\|_{H_1}$ for some $c \in [0, \infty)$. Note

$$|\langle Sx, y\rangle| = |h(x, y)| \leq \|h\|_{sesq}\|x\|_{H_1}\|y\|_{H_2} \quad \forall\ y \in H_2$$

and choosing $y = Sx$ we thus have

$$\|Sx\|^2 \leq \|h\|_s\|x\| \cdot \|Sx\| \implies \|Sx\| \leq \|h\|_s\|x\| \quad (\text{if } \|Sx\| = 0 \text{ then trivial})$$

and thus $\|S\|_{op} \leq \|h\|_s$.

"Norm preserving." From RR Theorem (baby) we have $\|f_x\|_{op} = \|z_x\| = \|Sx\|$ and since $\|f_x\|_{op} = \sup_{\|y\|_{H_2}=1} |f_x(y)| = \sup_{\|y\|=1} |h(x, y|$ we thus have

$$\|Sx\| = \sup_{\|y\|=1} |h(x, y)|$$

and then taking sup over $x \in H_1$ with $\|x\| = 1$ we thus have

$$\sup_{\|x\|=1} \|Sx\| = \sup_{\|x\|=1, \|y\|=1} |h(x, y)|$$

and the LHS is $\|S\|_{op}$ and the RHS is $\|h\|_{sesq}$.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Section 3.9. Hilbert-Adjoint Operator.

Hilbert-adjoint operator T^*. Let $T : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the *Hilbert-adjoint operator* T^* of T is the operator $T^* : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$ $\langle Tx, y\rangle = \langle x, T^*y\rangle$ and $\|T\| = \|T^*\|$.

Proof. Define $h : H_2 \times H_1 \to \mathbb{K}$ by $h(y, x) = \langle y, Tx\rangle$. h has a bounded sesquilinear form. Sesquilinearity is easy, to show boundedness see that

$$|h(y, x)| = |\langle y, Tx\rangle| \leq \|y\| \cdot \|Tx\| \leq \|y\| \cdot \|x\| \cdot \|T\|_{op}$$

and T is a bounded operator so $\|T\|_{op} < \infty$ verifies the boundedness of this sesquilinear form.

Thus by RR Theorem (adult), $\exists\ S : H_2 \to H_1$ defined by $\underbrace{h(y, x)}_{=\langle y, Tx\rangle_{H_2}} = \langle Sy, x\rangle_{H_1}$ so it seems a natural selection to take $T^* = S$.

Next see that

$$\|T^*\|_{op} = \|S\|_{op} = \|h\|_{sesq}$$

and we want to show that this is $\|T\|_{op}$. Thus similarly define $g : H_1 \times H_2 \to \mathbb{K}$ by $g(x, y) = \langle Tx, y\rangle \implies \exists!\ S : H_1 \to H_2$ such that $g(x, y) = \langle Sx, y\rangle$ and therefore $\langle Sx, y\rangle = \langle Tx, y\rangle \implies \|g\| = \|T\|$.

It is easy to see that $\|h\| = \|g\|$ by observing

$$\|g\|_{sesq} = \sup_{\|x\|=1, \|y\|=1} |\langle Tx, y\rangle| = \sup_{\|x\|=1, \|y\|=1} |\langle y, Tx\rangle| = \|h\|_{sesq}$$

verifying the norm preservation of the adjoint operator on a Hilbert space.

$\mathbb{Q.E.D.}$

Properties of Hilbert-adjoint operators. Let H_1, H_2 be Hilbert spaces, $S : H_1 \to H_2$ and $T : H_1 \to H_2$ bounded linear operators and α any scalar. Then we have

$$\begin{array}{rcl} \langle T^* y, x\rangle &=& \langle y, Tx\rangle \\ (S+T)^* &=& S^* + T^* \\ (\alpha T)^* &=& \bar{\alpha} T^* \\ (T^*)^* &=& T \\ \|T^*T\| &=& \|TT^*\| = \|T\|^* \\ T^*T = 0 &\iff& T = 0 \\ (ST)^* &=& T^* S^* \end{array}$$

Section 3.10. Self-Adjoint, Unitary and Normal Operators.

Self-adjoint, unitary and normal operators. A bounded linear operator $T : H \to H$ on a Hilbert space H is said to be

$$\begin{array}{rl} self-adjoint \text{ or } Hermitian \text{ if} & T^* = T \\ unitary \text{ if } T \text{ is bijective and} & TT^* = T^*T = I \\ normal \text{ if} & TT^* = T^*T \end{array}$$

Theorem (Self-adjointness). Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Then:

(a) T is self-adjoint $\implies \langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$

(b) H complex and $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H \implies T$ self-adjoint

Theorem (Sequences of self-adjoint operators). Let (T_n) be a sequence of bounded self-adjoint linear operators $T_n : H \to H$ on Hilbert H. Suppose (T_n) converges, $T_n \to T$ (i.e. $\|T_n - T\| \to 0$ where $\|\cdot\|$ is the norm on $B(H, H)$). Then T is also self-adjoint.

Section 4.2. Hahn-Banach Theorem.

Hahn-Banach Theorem (baby). X vector space over $\mathbb{K} = \mathbb{R}$, Z proper subspace of X. $f : Z \to \mathbb{R}$ is a linear functional such that $f \le p$ where p is sub-linear (i.e. $p(\alpha x) = \alpha p(x), \alpha \ge 0$ and $p(x+y) \le p(x) + p(y)$) $\implies \exists\ \bar{f} : X \to \mathbb{R}$ linear functional such that $\bar{f}\mid_Z = f$ and $\bar{f} \le p$.

Zorn's Lemma. M partially ordered ($\le$) set, i.e. (1) $a \le a$, (2) $a \le b, b \le a \implies a = b$, and (3) $a \le b, b \le c \implies a \le c$, and any chain (totally ordered subset) has an upper bound $\implies \exists$ maximal element in M.

Proof (HB baby). Define

$$M = \{g : \mathcal{D}(g) \to \mathbb{R} \mid g \text{ is linear functional}, Z \subseteq \mathcal{D}(g) \subseteq X, g\mid_Z = f, g \leq p\}$$

This is a partially ordered set under the ordering of $g_1 \leq g_2 \iff \mathcal{D}(g_1) \subseteq \mathcal{D}(g_2)$ and $g_2\mid_{\mathcal{D}(g_1)} = g_1$. Any chain $C \subseteq M$ has an upper bound given by

$$\hat{g}(x) = g(x) \text{ if } x \in \mathcal{D}(g) \quad \text{for any } g \in C$$

which is clearly a linear functional with domain

$$\mathcal{D}(\hat{g}) = \bigcup_{g \in C} \mathcal{D}(g)$$

Clearly $\hat{g}$ is an upper bound since by definition and construction we have $g \leq \hat{g}$ for all $g \in C$. Then there exists a maximal element $\bar{f}$ in M satisfying $\bar{f} \leq p$ and $\bar{f}\mid_Z = f$. We want to show that $\mathcal{D}(\bar{f}) = X$. We have that $\mathcal{D}(\bar{f}) \subseteq X$ so we must show that $\mathcal{D}(\bar{f}) \supseteq X$. For contradiction assume that latter does not hold.

Then $\exists\, y_1 \in X - \mathcal{D}(\bar{f})$ and consider $Y_1 = \text{span}(\mathcal{D}(\bar{f}), y_1)$. Note $y_1 \neq 0$ since $0 \in Z \subseteq \mathcal{D}(\bar{f})$ and $y_1 \in X - \mathcal{D}(\bar{f})$. Then for any $x \in Y_1$ we have $x = y + \alpha y_1$ for some $y \in \mathcal{D}(\bar{f})$. Note that this representation must be unique as if we have $x = y' + \alpha' y_1$ then $y' + \alpha' y_1 = y + \alpha y_1 \iff y - y' = (\alpha' - \alpha) y_1$ and the LHS is in $\mathcal{D}(\bar{f})$ and thus since $y_1 \notin \mathcal{D}(\bar{f})$ then $\alpha' - \alpha = 0 \implies \alpha' = \alpha$ and thus $y = y'$ showing this representation is unique.

Thus define g_1 on Y_1 by $g_1(y + \alpha y_1) = \bar{f}(y) + \alpha c$ where $c \in \mathbb{R}$. Clearly this is linear. For $\alpha = 0$ then $g_1 = \bar{f}$. Then g_1 is a proper extension of $\bar{f}$, contradicting the maximality of $\bar{f}$ if $g_1 \leq p$. See that

$$g_1(x) = \bar{f}(x) + \alpha c \leq -\alpha p\left(-y_1 - \frac{1}{\alpha} y\right) = p(\alpha y_1 + y) = p(x)$$

providing our contradiction.

$\mathbb{Q.E.D.}$

Hahn-Banach Theorem (adult). Z is a subspace of vector space X, $f : Z \to \mathbb{K}$ is $\mathbb{K}$-linear functional and $|f| \leq p$ where p sub-linear ($p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}$) $\implies \exists\, \bar{f} : X \to \mathbb{K}$ is $\mathbb{K}$-linear functional such that $|\bar{f}| \leq p$ and $\bar{f}\mid_Z = f$.

Note. $p(0) = 0$ and $p(x) + p(x) = p(x) + p(-x) \geq p(x + (-x)) = p(0) = 0 \implies p(x) \geq 0$ for any $x \in X$.

Proof (HB adult). $f : Z \to \mathbb{C}$ such that $f(x) = f_1(x) + i f_2(x)$ where $f_1, f_2 : Z \to \mathbb{R}$ are linear functionals. Note that f_2 is uniquely determined by f_1 defined by $f_2(x) = -f_1(ix)$. We can show this by matching real parts in the following equalities

$$\begin{aligned} \underbrace{f(ix)}_{=} &= f_1(ix) + i f_2(ix) \\ i f(x) &= i f_1(x) - f_2(x) \end{aligned}$$

Therefore

$$f(x) = f_1(x) - i f_1(ix)$$

Use HB baby on f_1 to extend f_1 (note $|f_1| \leq |f| \leq p$) to $\bar{f}_1 : X \to \mathbb{R}$. Naturally define

$$\bar{f}(x) = \bar{f}_1(x) - i \bar{f}_1(ix)$$

which is trivially a $\mathbb{C}$-linear functional, clearly $\bar{f}\mid_Z = f$ and we need to show that $|\bar{f}| \leq p$. Note that for any $z \in \mathbb{C}$ we have $z = re^{i\theta}$ and thus $\bar{f}(x) = |\bar{f}(x)| e^{i\theta}$ and therefore

$$|\bar{f}(x)| = \bar{f}(x) e^{-i\theta} = \bar{f}(e^{-i\theta} x) = \bar{f}_1(e^{-i\theta} x) - i \bar{f}_1(i e^{-i\theta} x)$$

and since the LHS is a real number then the imaginary part of the RHS must be 0. Then

$$0 \leq |\bar{f}(x)| = \bar{f}_1(e^{-i\theta} x) = |\bar{f}_1(e^{-i\theta} x)| \leq p(e^{-i\theta} x) = |e^{-i\theta}| p(x) = p(x)$$

verifying the boundedness by the sub-linear functional.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

HB Application 1. Let Z be a subspace of X a normed vector space, $f : Z \to \mathbb{K}$ is a $\mathbb{K}$-linear functional and bounded $\implies \exists\ \bar{f} : X \to \mathbb{K}$, bounded $\mathbb{K}$-linear functional such that $\bar{f}\mid_Z = f$ and $\|\bar{f}\| = \|f\|$.

Proof. Define $p : X \to \mathbb{R}$ by $p(x) = \|f\| \cdot \|x\|$ which is clearly sub-linear (note $\|f\| < \infty$ and it exists since f is bounded). Use HB Theorem (adult) so then $\exists\ \bar{f} : X \to \mathbb{K}$ a $\mathbb{K}$-linear functional such that $\bar{f}\mid_Z = f$ and $|\bar{f}| \leq p$.

Bounded? Note $|\bar{f}(x)| \leq p(x) = \|f\| \cdot \|x\| \implies \|\bar{f}\| \leq \|f\| < \infty$.

Equality of norms? Note that we have $\leq$ above so we WTS $\geq$. See that

$$\|\bar{f}\| = \sup_{x \in X - \{0\}} \frac{|\bar{f}(x)|}{\|x\|} \geq \sup_{x \in Z - \{0\}} \frac{|\bar{f}(x)|}{\|x\|} = \sup_{x \in Z - \{0\}} \frac{|f(x)|}{\|x\|} = \|f\|$$

verifying the equality using the boundedness above.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

HB Application 2. X normed vector space, $x \in X$. X' is space of bounded linear functionals $f : X \to \mathbb{K}$ ($\mathbb{K}$-linear functional). Fix $x \in X$, then

$$\bar{x} : X' \to \mathbb{K} \text{ defined by } \bar{x}(f) = f(x)$$

Further, $\|x\|_X = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{\|f\|_{op}} = \|\bar{x}\|_{op}$.

Proof. Note $\|\bar{x}\|_{op} \leq \|x\|_X$ since $|f(x)| \leq \|f\| \cdot \|x\| \implies \frac{|f(x)|}{\|f\|_{op}} \leq \|x\| \implies \|\bar{x}\|_{op} = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{\|f\|_{op}} \leq \|x\|_X$.

Now we want to show that $\|\bar{x}\|_{op} \geq \|x\|_X$. Construct $Z = \text{span}\{x\} = \{\alpha x \mid \alpha \in \mathbb{K}\}$. Let $g : Z \to \mathbb{K}$ be such that $g(\alpha x) = \alpha \cdot \|x\|$ and it is easy to see this is a linear functional on Z that is bounded because $|g(\alpha x)| = |\alpha| \cdot \|x\| = \|\alpha x\| \implies \|g\| = \sup_{z \in Z} \frac{|g(z)|}{\|z\|} = \sup_{z = \alpha x \in Z - \{0\}} \frac{\|\alpha x\|}{\|\alpha x\|} = 1$ proving $\|g\| = 1$.

Thus $\exists\ \bar{g} : X \to \mathbb{K}$ is bounded linear functional such that $\bar{g}\mid_Z = g$ and $\|\bar{g}\| = \|g\| = 1$. Then

$$\|\bar{x}\|_{op} = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{\|f\|_{op}} \geq \frac{|\bar{g}(x)|}{\|\bar{g}\|} = \frac{\|x\|}{1} = \|x\|$$

Therefore $\|\bar{x}|_{op} \geq \|x\|_X$ verifying the equality.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Adjoint operator. $T : H_1 \to H_2$ are Hilbert spaces $\implies \exists\ T^* : H_2 \to H_1$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ and $\|T^*\| = \|T\|$.

HB Application 3 (Adjoint operator). $T : X \to Y$ where X, Y normed vector spaces and T is a bounded linear operator $\implies \exists\ T^x : Y' \to X'$ bounded linear operator with $\|T^x\| = \|T\|$.

Proof. Define $T^x : Y' \to X'$ by $g \in Y' \mapsto T^x g \in X'$ where $T^x g : X \to \mathbb{K}$ is defined by $(T^x g) = g(Tx)$. We want to check this is linear, bounded, and preserves the norm of T.

"Bounded." $|(T^x g)(x)| = |g(Tx)| \leq \|g\| \|Tx\| \leq \|g\| \|T\| \|x\|$ using the boundedness of g and T.

"Linear." Cleary $T^x g$ as a functional on X is linear from the linearity of T and g. T^x's linearity follows.

"Norm equality." First see that

$$\|T^x g\|_{op} = \sup_{x \in X - \{0\}} \frac{|(T^x g)(x)|}{\|x\|_X} = \sup_{x \in X - \{0\}} \frac{|g(Tx)|}{\|x\|} \leq \sup_{x \in X - \{0\}} \frac{\|g\| \cdot \|T\| \cdot \|x\|}{\|x\|} = \|g\| \cdot \|T\|$$

which verifies that $\|T^x\|_{op} \leq \|T\|_{op}$. Next wee that

$$\begin{aligned} \|Tx\|_Y &= \|\overline{Tx}\|_{op} = \sup_{f \in Y'-\{0\}} \frac{|(\overline{Tx})(f)|}{\|f\|_{op}} = \sup_{f \in Y'-\{0\}} \frac{\overbrace{|f(Tx)|}^{=|(T^x f)(x)|}}{\|f\|} \\ &\leq \sup_{f \in Y'-\{0\}} \frac{\|f\| \cdot \|T^x\| \cdot \|x\|}{\|f\|} = \|T^x\|\|x\| \end{aligned}$$

showing $\|T\|_{op} \leq \|T^x\|_{op}$. Equality follows.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Baire's Category Theorem. Any complete metric space X is of second category.

First category. $X = \bigcup_{i \in \mathbb{N}} A_i$ where all A_i are nowhere dense. I.e. $\bar{A}_i$ has no open subsets (i.e. "indiscrete structure").

Second category. (Not first category.) $X = \bigcup_{i \in \mathbb{N}} B_i \implies \exists\ B_{i_0}$ such that $B_{i_0} \supseteq$ some open set.

Proof (BC Thm). Assume X is a complete metrix space. Assume for contradiction that X is not of second category. That is, X is first category. Let $X = \bigcup_{i \in \mathbb{N}} A_i$ where each A_i is nowhere dense (we know we can write X as this by it being of first category). Note X is an open set so $X \not\subseteq A_1 \subseteq \bar{A}_1$. Thus:

- $\bar{A}_1 \not\supseteq X \implies (\bar{A}_1)^C \neq \emptyset$ and $(\bar{A}_1)^C$ is open $\implies \exists\ \epsilon_1 > 0, x_1 \in (\bar{A}_1)^C$ such that $K_1 = B(x_1, \epsilon_1) \subseteq (\bar{A}_1)^C$.
- $\bar{A}_2 \not\supseteq X \implies (\bar{A}_2)^C \neq \emptyset$ and $(\bar{A}_2)^C$ is open and further $B\left(x_1, \frac{\epsilon_1}{2}\right) \not\subseteq \bar{A}_2 \implies K_2 = B\left(x_1, \frac{\epsilon_1}{2}\right) \cap (\bar{A}_2)^C \neq \emptyset$ and is open $\implies \exists\ \epsilon_2 > 0, x_2 \in K_2$ such that $B(x_2, \epsilon_2) \subseteq K_2$ (note $\epsilon_2 \leq \frac{\epsilon_1}{2} \leq \epsilon_1$).
- $\vdots$
- $\bar{A}_{n+1} \not\supseteq X \implies (\bar{A}_{n+1})^C \neq \emptyset$ and is open and $K_{n+1} = B\left(x_n, \frac{\epsilon_n}{2}\right) \cap (\bar{A}_{n+1})^C \neq \emptyset$ and open $\implies \exists\ \epsilon_{n+1} > 0, x_{n+1} \in K_{n+1}$ such that $B(x_{n+1}, \epsilon_{n+1}) \subseteq K_{n+1}$ (note $\epsilon_n \leq \frac{\epsilon_{n+1}}{2}$).
- $\vdots$

Claim: $\{x_n\}_{n \in \mathbb{N}}$ are Cauchy. Note these "balls" are getting smaller and smaller, they form a nested sequence:

$$\cdots \subseteq B(x_{n+1}, \epsilon_{n+1}) \subseteq B(x_n, \epsilon_n) \subseteq \cdots$$

Assume $m \geq n \implies x_m \in B(x_m, \epsilon_m) \subseteq \cdots \subseteq B\left(x_n, \frac{\epsilon_n}{2}\right) \implies x_m \in B\left(x_n, \frac{\epsilon_n}{2}\right) \implies d(x_n, x_m) < \frac{\epsilon_n}{2} \leq \frac{\epsilon_1}{2^n} \to 0$ since $\epsilon_n \leq \frac{\epsilon_n}{2^{n-1}}$ as $n \uparrow \infty$. Thus $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy and thus converges, $x_n \to x \in X$.

Now using $d(x_n, x_m) < \frac{\epsilon_n}{2} \implies$ taking $m \uparrow \infty$ we have $d(x_n, x_m) \to d(x_n, x) \leq \frac{\epsilon_n}{2} \implies x \in B(x_n, \epsilon_n) \subseteq (\bar{A}_n)^C$ by construction $\implies x \in (\bar{A}_n)^C$ for all $n \in \mathbb{N} \implies x \in \bigcap_{n \in \mathbb{N}} (\bar{A}_n)^C = \left(\underbrace{\bigcup_{n \in \mathbb{N}} \bar{A}_n}_{=X} \right)^C = X^C = \emptyset \implies x \in \emptyset$ gives our contradiction.

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

BC Theorem Application. T_n bounded linear operator, $T_n : X \to Y$ where X Banach and Y normed vector space $\implies \sup_{n\in\mathbb{N}} \|T_n\|_{op} < \infty$.

Proof. T_n bounded $\implies \|T_n\| < \infty$. Let $x \in X \implies \exists\ k \in \mathbb{N} \ni \sup_{n\in\mathbb{N}} \|T_n x\| \leq k$. Then for arbitrary $k \in \mathbb{N}$ we have

$$A_k = \left\{ x \in X \,\middle|\, \sup_{n\in\mathbb{N}} \|T_n x\| \leq k \right\} \implies x \in \bigcup_{k\in\mathbb{N}} A_k$$

and thus $X = \bigcup_{k\in\mathbb{N}} A_k$ and since X is of second category, $\exists\ k_0 \in \mathbb{N}$ such that A_{k_0} is nowhere dense. That is, $A_{k_0} \supseteq$ an open set $\implies B(x_0, \epsilon_0) \subseteq A_{k_0} \implies \|x - x_0\| < \epsilon_0 \implies x \in A_{k_0} \implies \sup_{n\in\mathbb{N}} \|T_n x\| \leq k_0$.

Using $\|T_n\|_{op} = \sup_{u\in S_X(0,1)} \|T_n u\|_X$. Note that for $u \in S_X(0,1)$ and $\epsilon < \epsilon_0$ we have $x_0 + \epsilon u \in B(x_0, \epsilon_0) \subseteq A_{k_0} \implies \sup_{n\in\mathbb{N}} \|T_n(x_0 + \epsilon u)\|_Y \leq k_0$ and thus

$$\epsilon\|T_n u\| - \|T_n x_0\| \leq \|T_n x_0 + \epsilon T_n u\|_Y \leq k_0 \implies \epsilon\|T_n u\| \leq k_0 + \|T_n x_0\| \leq k_0 + k_0 = 2k_0$$

and therefore $\|T_n u\| \leq \frac{2k_0}{\epsilon}$ and since k_0, ϵ were fixed then

$$\sup_{u\in S_X(0,1)} \|T_n u\| \leq \frac{2k_0}{\epsilon} \implies \|T_n\|_{op} \leq \frac{2k_0}{\epsilon} \implies \sup_{n\in\mathbb{N}} \|T_n\|_{op} \leq \frac{2k_0}{\epsilon}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$