MA 513
Test 2 Study Guide

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General

Logarithmic - $f(z) = \log z = \ln|z| + \arg z$ and $f(z) = \text{Log} z = \ln|z| + \text{Arg} z$

Trigonometric Functions - $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\tan z = \frac{\sin z}{\cos z}$

Hyperbolic Functions - $\sinh z = \frac{e^{z}-e^{-z}}{2}$ and $\cosh z = \frac{e^{z}+e^{-z}}{2}$

Length of Path - $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_a^b |z'(t)|\, dt$

Line Integral - $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$

Theorem on Bound of Integral - C is a contour of length L and f is a piecewise continuous function on $\mathbb{C}$. If we assume $|f(z)| \leq M \,\forall\, z \in \mathbb{C}$, then

$$\left|\int_C f(z)\right| \leq M \cdot L$$

Cauchy-Goursat Theorem - If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0$$

Cauchy Integral Theorem - f is analytic everywhere inside and on simple closed contour C, in positive sense. If z_0 is interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

and this can be extended to

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz, \qquad n = 1, 2, \ldots$$

Theorem - If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Series

Taylor Series Theorem - An analytic function f throughout a disk $|z - z_0| < R_0$ centered at z_0 and with radius R_0 has a unique power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad , \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Common Series

Good formulas to know:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ \cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \qquad (|z| < 1) \end{aligned}$$

Laurent Series Theorem - Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2 \ldots) \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots) \end{aligned}$$

Residues and Poles

Cauchy's Residue Theorem - Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Residue at Infinity - Residue at infinity is given by

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

and we can use this in the formula

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Residue Theorem 1 - An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic and nonzero and z_0. Moreover,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \qquad \text{if } m = 1$$

and

$$\operatorname*{Res}_{z=z_0} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \geq 2$$

Residue Theorem 2 - Let two function p and q be analytic at a point z_0. If

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Applications of Residues

Cauchy Principal Value - is given by

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

Evaluation of Improper Integrals

Steps to evaluate an integral $\int_0^\infty f(x)dx$ where f is even:

1. Draw a contour from $(-R, 0)$ to $(R, 0)$ (to the right) and then a semi-circle from $(R, 0)$ to $(-R, 0)$ counter-clockwise.

2. This is a closed contour and we can write

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=0}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

   where each z_k $(k = 0, 1, \ldots, n)$ are isolated singularities in the upper half-plane.

3. Look at when $|z| = R$ and show that $|f(z)|$ is bounded by M_R. Use this to show that

$$\left| \int_{C_R} f(z)dz \right| \leq M_R \cdot \pi R \to 0 \implies \int_{C_R} f(z)dz \to 0 \quad \text{as} \quad R \to \infty$$

4. Let $R \to \infty$ in 2. above and thus we have shown that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

   and using that f is even we see

$$\int_0^{\infty} f(x)dx = \pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Evaluation of Improper Integrals Using Indented Paths

Jordan's Lemma - Suppose that

- a function $f(z)$ is analytic at all points in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$

- C_R denotes a semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$, where $R > R_0$
- for all points z on C_R there is a positive constant $M_R$such that

$$|f(z)| \leq M_R \to 0 \quad \text{as } R \to \infty$$

Then for every positive constant a,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

Indented Paths - Use when $f(z)$ is not analytic at $z = 0$ and use the fact that

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)dz = -\pi i \cdot \operatorname*{Res}_{z=0} f(z)$$

and also use when $f(z)$ involving $\log z$ yet just show

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)dz = 0$$

Definite Integrals Involving Sines and Cosines

Evaluating integrals such as

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$$

can be done by looking at the circle $|z| = 1$ and converting this integral to an integral about that contour using the substitutions

$$\sin\theta = \frac{z - z^{-1}}{2i}, \qquad \cos\theta = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz}$$

Rouche's Theorem

Let C denote a simple closed contour and suppose that

- two functions $f(z)$ and $g(z)$ are analytic inside C
- $|f(z)| \geq |g(z)|$ at each point on C

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.